On Preservation Properties and a Special Algebraic Characterization of Some Stronger Forms of the Noetherian Condition

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Abstract. We give an elementary proof prove of the preservation of the Noetherian condition for commutative rings with unity R having at least one finitely generated ideal I such that the quotient ring is again finitely generated, and R is $I-$adically complete. Moreover, we offer as a direct corollary a new elementary proof of the fact that if a ring is Noetherian then the corresponding ring of formal power series in finitely many variables is Noetherian. In addition, we give a counterexample showing that the 'completion' condition cannot be avoided on the former theorem. Lastly, we give a elementary characterization of Noetherian commutative rings that can be decomposed as a finite direct product of fields.

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Introduction

Among the most studied rings in commutative algebra are the Noetherian ones, i.e., commutative rings with unity such that every ideal can be finitely generated. Moreover, one needs to increase a little bit the level of formal sophistication in order to find simple examples of non-Noetherian structures such as the ring of polynomials in countable many variables over a field or the ring of algebraic integers. So, (elementary) results preserving and implying the Noetherian condition after the application of standard algebraic operations (such as completion, quotient, localization, etc.) are quite useful (see (9), (4), (11) and (8)). Similarly, new characterizations of (stronger forms of the) Noetherian condition are oft very valuable for enlightening our understanding of what finite generation in (non-)commutative algebra means (see (14), (13), (2) and (7)).

Finally, we will prove in an elementary way two results concerning, on the one hand, the preservation of the Noetherian condition for a commutative ring with unity R having at least one finitely generated ideal I such that the quotient ring is again finitely generated and R is $I-$adically complete. In addition, we also offer as a corollary a new elementary proof of the fact that the ring of formal power series in finitely many variables is Noetherian, if its ring of coefficient so is, and we give a counterexample showing that the 'completion' condition cannot be avoided. On the other hand, we give a quite simple algebraic characterization of Noetherian commutative rings that can be decomposed as a finite direct product of fields.

1 Preservation Properties of the Noetherian Condition involving a Special Class of Finitely Generated Ideals

In this section, by a ring we will mean a commutative ring with identity, not necessarily Noetherian.

Let R be a ring, and let I be an ideal of R. We will show that if I is finitely generated and R/I is Noetherian, then the completion of R with respect to the I-adic topology is also Noetherian. In general one cannot expect R to be Noetherian under these hypothesis, as shown in Example 1 below. This result can be regarded as a generalization of the following well known corollary to the Cohen Structure Theorems ((3)), ((5, pag. 189,201)): if (R, m) is a quasilocal complete ring (with respect to the m-adical topology) then R is Noetherian if m is finitely generated. The Noetherian property is deduced from the fact that under these hypothesis R is a quotient of a power series ring over a complete discrete valuation ring. We observe that this result can be recovered immediately from Theorem 1 as the very special case when I is maximal, since R/I is a field, hence automatically Noetherian. While Cohen Structure Theorems require some machinery, the result below is totally elementary.

As a corollary, we deduce a quite elementary new proof of the fact that If A is Noetherian, so it is the power series ring $A[[x_1, \ldots, x_n]]$.

Theorem 1. *Let R be a ring, and let I be a finitely generated ideal of R. Suppose that R is I-adically complete, and that R/I is Noetherian. Then R is also Noetherian.*

Proof. If R were not Noetherian, a standard argument using Zorn's Lemma shows that there is a maximal ideal P in R with respect to the property of not being finitely generated, and this ideal is necessarily prime. It is clear that

P does not contain I, otherwise, since R/I is Noetherian, any lifting of a set of generator for P/I coupled with generators of I would also generate P. Thus, we may assume that there exist $z \in I$ such that $z \notin P$. By the maximality of P, the ideal $P + Rz$ must be finitely generated. Let $f_i + r_i z$, $i = 1, \ldots, n$ be any set of generators, with $f_i \in P$.

We claim that $\{f_1, \ldots, f_n\}$ is a set of generators for P. Let f be any element of P. Then, since f is a priory in $P + Rz$ there must be elements $g_i^{(0)} \in R$ such that

$$f = \sum_{i=1}^{n} g_i^{(0)}(f_i + r_i z) = \sum_{i=1}^{n} g_i^{(0)} f_i + z f^{(1)}, \tag{1}$$

where $f^{(1)} = \sum_{i=1}^{n} g_i^{(0)} r_i$. Hence, $f - \sum_{i=1}^{n} g_i^{(0)} f_i = z f^{(1)}$ is in P. Since we are assuming $z \notin P$, and P is prime we deduce $f^{(1)} \in P$.

The same reasoning applied to $f^{(1)}$ yields $f^{(1)} = \sum i = 1^n g_i^{(1)} f_i + z f^{(2)}$, for some elements $g_i^{(1)} \in R$, and $f^{(2)} \in P$. Replacing $f^{(1)}$ in 1 by the right hand side of this last equation gives $f = \sum_{i=1}^{n}(g_i^{(0)} + z g_i^{(1)}) f_i + z^2 f^{(2)}$. Since $z^t \notin P$ for any $t > 0$, a straightforward induction shows that f can be written as

$$f = \sum_{i=1}^{n}(g_i^{(0)} + z g_i^{(1)} + \cdots + z^t g_i^{(t)}) f_i + z^{t+1} f^{(t+1)}, \tag{2}$$

for certain elements $g_i^{(t)} \in R$, and $f^{(t+1)} \in P$. Since R is complete, $h_i = \sum_{t=0}^{\infty} z^t g_i^{(t)}$ is a well defined element of R.

We now observe that R must be I-adically separated, i.e., $\bigcap_{t=0}^{\infty} I^t$ is just the kernel of the canonical map $i : R \longrightarrow R^{\hat{I}}$, which should be the zero ideal, because i is an isomorphism, since we are making the assumption that R is I-adically complete. From this, we deduce that $f = \sum_{i=1}^{n} h_i f_i$, since

$$f - \sum_{i=1}^{n} h_i f_i = f - \sum_{i=1}^{n}(g_i^{(0)} + z g_i^{(1)} + \cdots + z^t g_i^{(t)}) f_i - z^{t+1} \sum_{i=1}^{n} \left(\sum_{j=0}^{\infty} z^j g_i^{(j+t+1)} \right) f_i$$

$$= z^{t+1} \left(f^{(t+1)} - \sum_{i=1}^{n} \left(\sum_{j=0}^{\infty} z^j g_i^{(j+t+1)} \right) f_i \right),$$

is an element of I^t, for all t.

In conclusion, any element $f \in P$ can be generated by the set $\{f_1, \cdots, f_n\} \subseteq P$. Therefore, P would be finitely generated, which contradicts our former assumption.

Corollary 1. *If A is Noetherian, so it is $A[[x_1, \ldots, x_n]]$.*

Proof. Let $R = A[[x_1, \ldots, x_n]]$ and $I = (x_1, \ldots, x_n)$. A standard argument ((5, pag. 192)), ((10, pag. 61)) shows that $R = B^{\hat{I}}$ where $B = A[x_1, \ldots, x_n]$. Since $R/IR \simeq A$ is Noetherian, the previous theorem gives that so it is R.

In general, if R is not complete with respect to the I-adic topology, it is not true that R is Noetherian under the hypothesis of I being finitely generated and R/I being Noetherian, not even in the case where I is maximal, as the following example shows.

Example 1. Let $\mathbb{N}$ denote the set of natural numbers, and let U be any non principal ultrafilter in N, that is, a collection of infinite subsets of N, closed under finite intersection, with the property that for any $D \subset N$, either D or its complement belongs to U. Let (R, m) be any discrete valuation ring, and let us denote by R_w a copy of R indexed by the natural $w \in N$. By S we will denote the ultraproduct, $S = u \lim_{w \to \infty} R_w$. We recall that this is defined as the set of equivalent classes in the Cartesian product $\prod_{w \in \mathbb{N}} R_w$, where two sequences (a_w) and (b_w) are regarded as equivalent if the set of indices w where $a_w = b_w$ is an element of U. This is a ring with the obvious operations, and it is also local with a principal maximal ideal m' generated by the class of (p_w), where $m_w = (p_w)$ is the maximal ideal of R_w (see (12, Ch. 1-2)). If c denotes the class of the sequence of powers $(p_w)^w$, then it is clear that c belongs to the Jacobson radical of S, $\cap_{w=0}^{\infty} (m')^w$, and it is a nonzero element. Consequently, S cannot be Noetherian, even though its maximal ideal is finitely generated (actually, principal), and S/m' is a field.

2 A Characterization of a Noetherian Finite Direct Product of Fields

In this section we will give an elementary algebraic re-formulation of the fact that a commutative Noetherian ring is the direct product of fields by means of a idempotent-membership condition, namely, the fact that any of its elements belongs lo the ideal generated by its-own square.

Theorem 2. *Let R be a commutative Noetherian ring. Then R is the finite direct product of fields if and only if any element $f \in R$, holds that $f \in (f^2)$.*

Proof. If R is a finite product of fields, then clearly the desired condition is satisfied, since any element in R is the direct product of zeros and unities.

Conversely, let us assume, by contradiction, that R is a Noetherian ring which is not a finite product of fields. We want to prove that there is an element $f \in R$ such that $f \notin (f^2)$. In fact, we can reduced to the case of R connected, because if $\mathrm{Spec} R$ is not connected then, due to the Noetherian hypothesis, we can write $\mathrm{Spec} R = V(Q_1) \uplus \cdots \uplus V(Q_s)$, where $V(Q_j) \cong \mathrm{Spec}(R/Q_j)$ are the connected components of $\mathrm{Spec} R$. Hence, by the Chinese Remainder Theorem (1), $R \cong \prod_{i=1}^{s} R/Q_i$ and by the previous assumption at least one of the R/Q_i is not a field. So, it is enough to find an $f_i \in R/Q_i$ such that $f_i \notin (f_i^2)$ to obtain the desired element $f = (0, ..., f_i, ..., 0) \in R$. Now, the connectedness of $\mathrm{Spec} R$ it is equivalent to saying that the only idempotents of R are trivial ones, namely, zero and one (see for example (6, Ch. 2)).

Lastly, choose $f \in R$ neither a unit nor idempotent. Then, $f \notin (f^2)$. In fact, by contradiction, if $f = cf^2$, for some $c \in R$, and so $cf(1 - cf) = 0$, which means that cf is idempotent. Hence, $cf = 0$ or $cf = 1$. In the first case we have $f = (cf)f = 0$, and in the second case, f is a unit. Then both cases contradicts our hypothesis on f.

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