ON EXTENSIONS OF DIRICHLET AND GREEN-TAO THEOREMS AND GOLDBACH-DIRICHLET REPRESENTATIONS OVER CERTAIN FAMILIES OF COMMUTATIVE RINGS WITH UNITY

DANNY A. J. GÓMEZ-RAMÍREZ AND ALBERTO F. BOIX

ABSTRACT. In this paper, we study stronger forms of Goldbach's conjecture enriched with the linear representations of prime numbers given by the classical Dirichlet theorem and its extensions. We call such a representation a Goldbach-Dirichlet representation (GD-representation). Among other results, we show that Dirichlet's Theorem on Arithmetic Progressions is, in general, not true in the ring of formal power series over the integers. Additionally, we use a polynomial version of the Schinzel hypothesis due to A. Bodin, P. Dèbes and S. Najib to prove the existence of GD-representations for a wide collection of polynomial rings over special families of fields of characteristic zero, among others. Moreover, we study the (non)validity of Dirichlet's Theorem over several families of commutative rings with unity like polynomial and formal series rings. Finally, we obtain a generalization for polyomial rings of the celebrated Green–Tao Theorem.

INTRODUCTION

Motivated by the heuristic and multidisciplinary principles of the New Cognitive-Computational Foundations' of Mathematics Program as the first basic pillar of Cognitive-Computational Metamathematics (CCMM) or Artificial Mathematical Intelligence (AMI) [GR20], and following similar methodological principles of our previous work [BGR]; we study seminal problems in classic number theory like Dirichlet's and Green-Tao's theorems on arithmetic progressions, and Goldbach's conjecture from an extended perspective. In fact, we develop a sort of bottom-up heuristic approach for extending and enriching these results to very similar, although in some aspects significantly different, algebraic-arithmetic structures in comparison with the standard ring of usage in number theory: the integers. Even more, we obtain a lot of motivation by the initial global taxonomy of fundamental metamathematical cognitive mechanisms that use our minds responsible for our abstract-formal creativity and immersed in the whole multidimensional goal of CCMM [GR20,

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Part II]. In particular, our working perspective for refining, extending and enlightening the aforementioned problems (and, in general, any mathematical inquiry) is not simply starting by solving particular cases of the initial puzzle with an increased generality, but changing slightly both the original inquiry as well as the ground algebraic structure involved. Following these lines we were able to find a wide spectrum of innovative extensions, some of them computationally feasible, of Goldbach's conjecture [BGR].

Most of the results described in this paper can be seen as a form of extended Diophantine scrutiny for formal polynomial-like structures in the following sense. Classic Diophantine analysis is understood as resolving equations of the form $P(\mathbf{x}) = 0$ for $P(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$, i.e., where the variables $\mathbf{x} \in \mathbb{Z}^n$. Equivalently, its core purpose is to find solutions of the relation $P(\mathbf{x}) \in \{0\}$ for $\mathbf{x} \in \mathbb{Z}^n$ (or, for some suitable subset of $\mathbb{Z}^n$). So, if we allow to change the set $\{0\}$ by other sets with highly arithmetical interest, like, for example, the set $\mathbb{P}_R$ of irreducible (e.g., prime) elements of the ring in consideration (typically $R = \mathbb{Z}$), then we see that a lot of classic results and conjectures in (classic) Number Theory like Schinzel's hypothesis H [SS58], Landau's conjecture [Pin09], and Dirichlet's theorem for primes in arithmetic progressions [Kha22, Theorem 10.10], among others, can be formally seen as an extended part of Diophantine analysis.

One of the main goals of this paper is to study (non)extensions of Dirichlet's theorem on primes in arithmetic progressions for additional rings of interest. Recall that the classic form of Dirichlet's theorem for primes in arithmetic progressions asserts [Kha22, Chapter 10, Theorem 10.10] that if a and b are relatively prime positive integers, then there are infinitely many primes in the arithmetic progression $a(-) + b$ running over the positive integers. The first general question we want to partly tackle in this paper is the following.

Question 0.1. *Let R be a commutative unique factorization domain (UFD), and let $a,\ b \in R$ be coprime elements; in other words, they satisfy*

$$Ra \cap Rb = Rab.$$

Under which conditions we can guarantee that the arithmetic progression $aX + b$ contains infinitely many prime elements of R?

So, following the methodological heuristics of Cognitive-Computational Metamathematics (CCMM) or Artificial Mathematical Intelligence [GR20], we see that two algebraic structures of interest for studying Question 0.1 would be $\mathbb{Z}[x]$ and $\mathbb{Z}[\![x]\!]$. For more details along these methodological lines, the reader can see the introduction of [BGR] to get a deeper idea about the main heuristic principles coming from CCMM in the context of studying suitable and enlightening extensions of Goldbach's conjecture.

One of the main results of this paper (see Theorem 3.3) is to provide a counterexample to Question 0.1 over the ring $\mathbb{Z}[\![x]\!]$. This does not come completely as a surprise for us, mainly because it has been recently proved by E. Paran in [Par20] that Goldbach's conjecture does not hold in general over $\mathbb{Z}[\![x]\!]$. Both results show, roughly speaking, that the rings $\mathbb{Z}$ and $\mathbb{Z}[\![x]\!]$ have a quite different arithmetic nature.

In general, in the literature one has studied a lot properties of $\mathbb{Z}[\![x]\!]$ involving concepts of commutative algebra like Krull dimension, primal ideals, UFD-s, among others. However, regarding $\mathbb{Z}[\![x]\!]$ there are no so much results about subtle arithmetical properties like the ones coming from elementary number theory and involving extensions of classic problems like Dirichlet's theorem, etc. which are one of the main goals of this paper.

Another beautiful recent result about primes in arithmetic progressions is the so–called Green–Tao's Theorem, obtained by Green and Tao in [GT08, Theorem 1.1]. It asserts that, given any $k \in \mathbb{N}$, there are coprime integers a and b such that all the numbers $a+b,\ a+2b, \ldots,\ a+kb$ are primes. The second general question we want to tackle in this paper is the following one.

Question 0.2. *Let R be a commutative unique factorization domain, let $r \in R$ and an integer $k \geqslant 1$. Under which assumptions we can guarantee the existence of coprime elements $a,\ b \in R$ such that all $a + r \cdot b,\ a + 2r \cdot b, \ldots, a + kr \cdot b$ are prime elements of R?*

The second goal of this paper is to extend the Green–Tao's Theorem for polynomial rings in several variables with coefficients either integers or in a field of characteristic zero, providing a partial positive answer to Question 0.2. We refer to Theorem 5.2 for the precise statement. Concerning Question 0.2, and to the best of our knowledge, the only attempt to partially tackle it has been done in [GOS23].

Now, we provide a brief outline of the contents of this paper for the convenience of the reader. First of all, in Section 1, after recalling some notations and facts we plan to use along the paper, our main goal will be to introduce the so–called *Goldbach–Dirichlet Representations* (see Definition 1.4) which will play a crucial role in Section 4 of this paper. Secondly, in Section 2 we provide results analogous to Dirichlet's theorem for primes in arithmetic progressions, on the one hand, for the polynomial ring in several variables over the integers (see Theorem 2.1) and, on the other hand, for some rings of integers of number fields (see Proposition 2.2). Section 3 contains one of the main results of this paper (see Theorem 3.3); namely, a counterexample to Dirichlet's theorem for primes in arithmetic progressions over the ring $\mathbb{Z}[\![x]\!]$. In Section 4, building upon the celebrated results obtained by Bodin, Dèbes and Najib in [BDN20], we show one of the most surprising results in this paper, the existence of Goldbach–Dirichlet representations for polynomial rings, which can be seen as a stronger and global form of the Chinese Remainder

Theorem for certain kinds of polynomial rings. Finally, in Section 5 we obtain the third main result of this paper (see Theorem 5.2); namely a extended and 'kind of irregular form 'of Green–Tao theorem for a polynomial ring in several variables over a field of characteristic zero.

1. Preliminaries

The goal of this section is, on the one hand, to establish some results that will be used along the rest of the paper and, on the other hand, to introduce the notion of Goldbach–Dirichlet's representation that will play a key role along the paper, overall in Section 4.

The first fact we want to single out is the following one.

Remark 1.1. *Let R be an integral domain (not necessarily factorial), and let $a,\ b \in R$ be coprime elements; in other words, they satisfy*

$$Ra \cap Rb = Rab.$$

Under these assumptions, it is known [Fos73, Lemma 14.1] *that $bX - a$ is a prime element of $R[X]$.*

Now, we want to review a polynomial form of Schinzel's hypothesis for some polynomial rings proved in [BDN20, Theorem 1.1. and Lemma 2.1]. Let us explicitly state the form of the result that we need here.

Theorem 1.2. *Let R be either $\mathbb{Z}$ or $\mathbb{K}[\mathbf{z}] := \mathbb{K}[z_1, \ldots, z_m]$ the ring of polynomial on several variables over an arbitrary field $\mathbb{K}$ (or more generally, let R be an infinite UFD with fraction field $\mathbb{F}$ with the product formula, or imperfect ($\mathbb{F}^p \neq \mathbb{F}$) in the positive characteristic case). Let $S = R[\mathbf{x}] := \mathbb{K}[x_1, \ldots, x_n]$, for $n \geqslant 1$, and let $P_1, \ldots, P_w \in R[\mathbf{x}, y] = S[y]$ be irreducible polynomials of degree $\geqslant 1$ in y.*

Then, there exist $M \in S$ of arbitrarily large degree in each of the variables such that $P_1(\mathbf{x}, M(\mathbf{x})), \ldots, P_w(\mathbf{x}, M(\mathbf{x}))$ are all irreducible in S.

Remark 1.3. *In Theorem 1.2, and for the purposes of generality of our presentation, we can think of R being either $\mathbb{Z}$ or a polynomial ring in several variables over a field of characteristic zero. Note that under the hypothesis of Theorem 1.2 it is not necessary to impose the analogue version of the classic (technical) condition of Schinzel conjecture, (i.e., no irreducible (prime) polynomial $Q \in S$ divide the product $\prod_{n=1}^{w} P_n(\mathbf{x}, M(\mathbf{x}))$ for all $M(\mathbf{x}) \in S$, since this happens in the context of the given hypothesis due to* [BDN20, Lemma 2.1].

1.1. Goldbach-Dirichlet Representations.

The goal of this part is to introduce the notion of Goldbach–Dirichlet representations over unique factorization domains. Roughly speaking, our goal is to find rings where Goldbach's conjecture can be achieved using just primes in arithmetic progressions. The precise statement is formulated as follows.

Definition 1.4. *Let S be a unique factorization domain (UFD). We say that S has* Goldbach-Dirichlet additive representations of length $n+1$, *(or, simply,* GD-representations*) for some fixed $n \in \mathbb{N}$, if for any $Q \in S$, and any $a_0, a_1, \ldots, a_n, \in S$, $b_0, b_1, \ldots, b_n \in S$ such that the following two seminal conditions hold:*

(i) $\gcd(a_i, b_i) = 1$ *for all* $i = 1, \ldots, n$.
(ii) $\gcd(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i - Q) = 1$.
(iii) *(a_0 is a unit in S. In the case that this additional condition holds, we say that S has special* Goldbach-Dirichlet additive representations of length $n+1$, *or, simply,* SGD-representations*)*

Then, Q can be represented as the sum of $n+1$ prime elements of S, $p_i = a_i y + b_i$, for a (global) $y \in S$. In other words, there exists a special additive representation of Q of the form

$$Q = \sum_{i=0}^{n} (a_i y + b_i), \text{ where each term } a_i y + b_i \text{ is prime in } S. \tag{1.1}$$

If Q can be represented as the sum of $n+1$ prime elements of S, $p_i = a_i y_i + b_i$, for some (possibly different) $y_i \in S$, the we say that S has Local (Special) Goldbach-Dirichlet representations. (or simply L(S)GD-representations), i.e.,

$$Q = \sum_{i=0}^{n} (a_i y_i + b_i), \text{ where each term } a_i y_i + b_i \text{ is prime in } S. \tag{1.2}$$

Remark 1.5. *The reader will easily notice that condition (i) of Definition 1.4 implies that, for any $1 \leqslant i \leqslant n$, the polynomial $a_i X + b_i \in S[X]$ is irreducible. This is exactly what we review along Remark 1.1. By the same reason, condition (ii) of Definition 1.4 implies that the polynomial*

$$\left(\sum_{i=1}^{n} a_i \right) X + \left(\left(\sum_{i=1}^{n} b_i \right) - Q \right)$$

is irreducible.

2. Dirichlet's Theorem for polynomial rings over the integers and for some rings of integers of number fields

The goal of this section is to state and prove Dirichlet's type results for arithmetic progressions, on the one hand, over a polynomial ring in several variables over the integers and, on the other hand, over some rings of integers of number fields.

First of all, recall that it is known that the polynomial ring $\mathbb{Z}[x]$ is a unique factorization domain, see for instance [Fos73, Theorem 8.1 and Corollary 8.2].

The first result of this section is the following one, which can be regarded as a consequence of Theorem 1.2.

Theorem 2.1. *Let $S := \mathbb{Z}[x_1, \ldots, x_n]$, with $n \geqslant 1$ and let $a, b \in S$ be coprime elements, i.e., a and b have no common prime divisors. Then, there are infinitely many elements $w \in S$ such that $aw + b$ is prime in S.*

Proof. Set $R = \mathbb{Z}$, $S = R[x_1, \ldots, x_n]$ and $P_1(x_1, \ldots, x_n, y) := ay + b \in S[y]$ in Theorem 1.2. Since a and b are coprime by assumption, Remark 1.1 ensures that P_1 is irreducible in $S[y]$. Thus, by Theorem 1.2 there are infinitely many polynomials (of arbitrarily large degree on any fixed variable x_j) $w := M(x_1, \ldots, x_n)$ such that $aw + b$ is irreducible (prime) in the UFD S. □

Our second main result in this section is a Dirichlet's theorem for arithmetic progressions over the Gaussian integers and over the Eisenstein integers.

Proposition 2.2. *The following assertions hold.*

(i) Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers, and let p, q be integers that are Gaussian primes. Then, the arithmetic progression $px + q$ contains infinitely many integers that are Gaussian primes, where x runs over all the positive integers.

(ii) Let $R = \mathbb{Z}[\omega]$ be the ring of Eisenstein integers, and let p, q be prime integers that are Eisenstein primes. Then, the arithmetic progression $px+q$ contains infinitely many integers that are Eisenstein primes, where x runs over all the positive integers.

Proof. First of all, let p and q be integers that are Gaussian primes. It is known (see for instance [Neu99, Theorem (1.4)]) that both p and q are primes of the form $p = 4n + 3$, $q = 4m + 3$, $n \geqslant 1$, $m \geqslant 1$. Now, we observe that $px + q \equiv 3 \pmod 4$ if and only if $x = 4k$ for some integer $k \geqslant 1$. Therefore, we have that $px + q = 4pk + q$, and since $\gcd(4p, q) = 1$, we conclude, by appealing to Dirichlet's theorem on arithmetic progressions over $\mathbb{Z}$ and to [Neu99, Theorem 1.4], that the sequence $px + q$ contains infinitely many positive integers that are Gaussian primes.

Secondly, let p and q be integers that are Eisenstein primes. Then, it is known (see for instance [Cox22, Chapter 1, §4.A]) that both p and q are primes of the form $p = 3n + 2$, $q = 3m + 2$, $n \geqslant 1$, $m \geqslant 1$. Assume that x only can take positive integer values; in this case we have that $px + q \equiv 2 \pmod 3$ if and only if $x = 3k$ for some integer $k \geqslant 1$. Therefore, we have that $px + q = 3pk + q$, and since $\gcd(3p, q) = 1$, we conclude, by appealing to Dirichlet's theorem on arithmetic progressions over $\mathbb{Z}$ and to [Cox22, Ch 1. §4.A], that the sequence $px + q$ contains infinitely many positive integers that are Eisenstein primes. □

Remark 2.3. *Recall that, given $\beta \in \mathbb{Z}[i]$ that is not divisible by $1+i$, there are infinitely many Gaussian primes $\mathfrak{p}$ satisfying $\mathfrak{p} \equiv 1 \pmod{\beta}$, see* [LE23, Corollary 4.21].

Now, let K be a number field, and let $\mathbb{Z}_K$ be its ring of integers. Following standard terminology in Algebraic Number Theory, given $p \in \mathbb{Z}$ a prime number, we say that p is **inert in** K provided the principal ideal $p\mathbb{Z}_K$ is a prime ideal of $\mathbb{Z}_K$. On the one hand, it is known that if the Galois group of the field extension $K \mid \mathbb{Q}$ is not cyclic, then there are only finitely many inert primes in K, see [Neu99, Chapter I, §9, Exercise 1]. On the other hand, it is also known that, if the Galois group of $K \mid \mathbb{Q}$ is cyclic, then there are infinitely many inert primes in K, see for instance [Jan73, Chapter IV, Corollary 5.4]. Motivated by Proposition 2.2, we want to raise the following question.

Question 2.4. *Let K be a number field such that the Galois group of $K \mid \mathbb{Q}$ is cyclic, and let $p,\ q$ be prime numbers that are inert in K. Is it true that the arithmetic progression $px+q$ contains infinitely many inert primes in K, where x runs over the non–negative integers?*

Remark 2.5. *The question is even interesting in case of a quadratic field extension. Indeed, let D be a squarefree integer, let $K = \mathbb{Q}(\sqrt{D})$, and let $p,\ q$ be prime numbers that are inert in K. By* [Kha22, Theorem 4.11]*, we know that*

$$\left(\frac{D}{p}\right) = -1 = \left(\frac{D}{q}\right),$$

where $(-)$ denotes the Kronecker symbol. In other words, p and q are prime numbers not dividing D such that D is not a quadratic residue neither modulo p nor modulo q. However, for us it is not clear whether the arithmetic progression $px+q$ contains infinitely many prime numbers such that D is not a quadratic residue modulo $px+q$.

Remark 2.6. *To the best of our knowledge, one of the few places where arithmetic progressions over number fields have been considered is in* [FJ23]*. Indeed, given a number field K with ring of integers $\mathbb{Z}_K$, it is known* [FJ23, Theorem 14.3.6] *that, on the one hand, any separable Hilbert set H of K* [FJ23, pages 229–230] *contains an arithmetic progression and, on the other hand, that the intersection of any arithmetic progression with H is non–empty. However, the reader has to keep in mind that, according to* [FJ23, page 253]*, the authors define an arithmetic progression in $\mathbb{Z}_K$ as a set of the form $q+\mathfrak{a}$, where $q \in \mathbb{Z}_K$ and $\mathfrak{a} \subset \mathbb{Z}_K$ is an ideal. In this way, at least for us, it is not clear whether a separable Hilbert subset of K contains an arithmetic progression in our sense, where we only deal with $\mathfrak{a}$ a principal ideal.*

3. The non–validity of Dirichlet's Theorem in the ring of Formal Power Series over the Integers

The goal of this section is to show that Dirichlet's Theorem for primes in arithmetic progressions is in general not true over the ring of formal power series in one variable over the integers. We start by first recalling some known facts.

Indeed, it is known that the formal power series ring $\mathbb{Z}[\![x]\!]$ is a unique factorization domain, due to a theorem by P. Samuel [Sam61, Theorem 2.1 and Corollary 2.2], since $\mathbb{Z}$ is a PID (see also additional proof in [BG08, Theorem 3.8.] and [Fos73, Chapter V] for more general statements). So, in $\mathbb{Z}[\![x]\!]$ the notions of prime and irreducible element coincide.

Definition 3.1. *Let R be a commutative ring with unity, and $a, b \in R$. We say that a and b are essentially the same element (as factor) if they are associate, i.e., there exists a unit $u \in R$ such that $a = ub$. In particular, we say that a collection of elements $U \subseteq R$ has cardinality essentially ω if the cardinality of U/rel has cardinality ω, where rel is the equivalence relation given by being associate. So, if we say that a collection of elements U is essentially infinite, then it means that U has infinitely many elements (resp. finite) even when we consider two associates as the same element.*

The former definition is particularly useful when we consider a ring with infinitely many units. In other words, in this case, a set with infinitely many elements can have essentially finite elements, i.e., finite equivalence classes of certain sort of elements, where each class has infinitely many elements.

Remark 3.2. *Note that coprimality in the ring of formal power series over the integers is a more subtle condition than just verifying that the corresponding independent terms are coprime. Explicitly, keep in mind that a special class of prime element in $\mathbb{Z}[\![x]\!]$ are the series*

$$h = \sum_{i \geqslant 0} h_i x^i, \ h_i \in \mathbb{Z},$$

such that $h_0 = p^k$ for some prime p and a natural number $k > 0$, and h_1 not being divisible by p [BGW12, Proposition 2.1.(c)]. *So, for example, the elements $r = p + x$ and $s = p^2 + x$ are prime. Thus, r^2 and s both have as independent term p^2, however, both are coprime as formal series, because they are formed as the product of completely different primes, i.e., r and s, respectively. In fact, r and s cannot be associated because all the units in $\mathbb{Z}[\![x]\!]$ are exactly the series with independent term ± 1* [BG08, Proposition 3.1]. *So, the fact that two formal series a and b are coprime as series does not imply any similar concrete condition on the level of the independent terms a_0 and b_0.*

Indeed, as an example let $p \in \mathbb{Z}$ be a prime number. Note that in $\mathbb{Z}[\![x]\!]$ the elements $a = p + x$ and $b = p + 2x$ are different prime elements, since its

independent terms are primes. So, a and b are coprime. However, $a_0 = b_0 = p$ are not coprime numbers.

Regarding Dirichlet's theorem on primes in arithmetic progressions for $\mathbb{Z}[\![x]\!]$, we will show in the form of the next statement that there is a simple and elementary counterexample. However, in this setting there are much additional interesting properties to study about the (non-)validity of Dirichlet's theorem for sequences of the form $ah + b$, where $a, b \in \mathbb{Z}[\![x]\!]$ are fixed elements and h varies as formal series. These phenomena will be studied in more detail in Proposition 3.5.

Theorem 3.3 (Counterexample to an analogue for formal power series of Dirichlet's Theorem)**.** *Let $S := \mathbb{Z}[\![x]\!]$ and let $a = 34 + x \in S$ and $b = 6 \in S$. Then, a and b are coprime elements of S, and the sequence $ah + b$ varying $h \in S$ contains only non-irreducible elements. In other words, $ah + b$ is always a reducible element in S.*

Proof. Firstly, note that $a = 2 \times 17 + x$. So, following the proof of [BG08, Proposition 3.4], we see that there exist two formal series $p, q \in S$, such that $a = pq$, with $p_0 = 2$, and $q_0 = 17$. More explicitly, if

$$p(x) = \sum_{i \geqslant 0} p_i x^i, \; q(x) = \sum_{i \geqslant 0} q_i x^i,$$

then we have, since $1 = (-8) \times 2 + 1 \times 17$ is the Bézout identity for 2 and 17, that

$$p_0 = 2, \; p_1 = 1, \; p_j = -\sum_{k=1}^{j-1} p_k q_{j-k}, \; (j \geqslant 2),$$

$$q_0 = 17, \; q_1 = -8, \; q_j = 8\sum_{k=1}^{j-1} p_k q_{j-k}, \; (j \geqslant 2).$$

Now, by [BG08, Proposition 3.3] p and q are prime elements of S since their independent terms are primes in $\mathbb{Z}$. Due to the fact that the coefficient of x in a is 1, we verify directly that neither p nor q can be constant formal series. So, $a = p \times q$ is a prime decomposition of a in S. On the other hand, by [BG08, Proposition 3.2.] $b = 2 \times 3$ is the prime decomposition of b in S. Finally, note that 2 and p are non-associates, because otherwise 2 would divide all the coefficients of p, but $p_1 = 1$. Thus, since the former prime factors in S of a and b are all essentially different, we conclude that a and b are coprime in S.

For the last part, by [BG08, Proposition 3.4] it is enough to show that $a_0h_0 + b_0$ is neither zero nor a prime power for all $h_0 \in \mathbb{Z}$. In fact, since 17 does not divide 6, then $a_0h_0 + b_0 \neq 0$. Furthermore, note that $a_0h_0 + b_0 = 2(17h_0 + 3)$. So, $a_0h_0 + b_0$ is a prime power if and only if $17h_0 + 3$ is 1 or a power of 2. Clearly $17h_0 + 3 \neq 1$, so, it remains to show that $17h_0 + 3 \neq 2^k$, for any $k \geqslant 1$. Effectively, by direct computation we verify that the only possible

residues r of powers of 2 modulo 17 are $r = 1, 2, 4, 5, 8, 9, 13, 15$ and 16. On the other hand, $17h_0 + 3$ has always residue 3 modulo 17. So, $17h_0 + 3$ can never be a power of two, for any $h_0 \in \mathbb{Z}$. In conclusion $ah + b$ is never an irreducible element of S. This finishes our proof. □

Remark 3.4. *The counterexample given by Theorem 3.3 should be compared with Theorem 2.1. Indeed, once again setting $a = 34 + x$ and $b = 6$, we observe that, on the one hand, Theorem 2.1 ensures that the arithmetic progression $aY + b$ produces infinitely many irreducible elements of $\mathbb{Z}[x]$ when $Y \in \mathbb{Z}[x]$. However, on the other hand our counterexample shows that $aY + b$ produces no irreducible elements of $\mathbb{Z}[\![x]\!]$ when $Y \in \mathbb{Z}[\![x]\!]$.*

Theorem 3.3 shows that in $\mathbb{Z}[\![x]\!]$ the arithmetic progression $ay + b$ might contain no irreducible elements. In the next result, we explore in some detail how general is this phenomenon in this power series ring.

Proposition 3.5. *Let $S := \mathbb{Z}[\![x]\!]$ and let $a, b \in S\backslash\{0\}$ be coprime elements, i.e., a and b have no common prime divisors. Then, the following assertions hold.*

- *(i) If $\gcd(a_0, b_0) = 1$, then there are essentially infinitely many prime elements of the form $aw + b$ varying $w \in S$.*
- *(ii) If $\gcd(a_0, b_0) = d \neq 1$ is not a prime power, then there are essentially finitely many prime elements of the form $aw + b$ varying $w \in S$.*
- *(iii) If $\gcd(a_0, b_0) = p^m$, for some prime number p and some natural number $m > 0$; such that $p^{m+1} | a_0$ and $p^{m+1} \nmid b_0$, then there are essentially finitely many prime elements of the form $aw + b$ varying $w \in S$.*
- *(iv) If $\gcd(a_0, b_0) = p^m$, for some prime number p and some natural number $m > 0$; such that $p^{m+1} \nmid a_0$, then there could be essentially finitely many prime elements or infinitely many primes of the form $aw+b$ varying $w \in S$, depending of the particular values of $a_0, a_1, b_0.b_1$.*

Proof. Let $h = \sum_{i=0}^{\infty} h_i x^i \in \mathbb{Z}[\![x]\!]$ be a generic element. Set $a = \sum_{i=0}^{\infty} a_i x^i$ and $b = \sum_{i=0}^{\infty} b_i x^i$, where $a_i, b_i \in \mathbb{Z}$, for $i \in \mathbb{N}$.

(i) Assume that $\gcd(a_0, b_0) = 1$. Then, by the classic version of Dirichlet's theorem for primes in arithmetic progressions [Kha22, Theorem 10.10], there exists infinitely many $h_0 \in \mathbb{N}$ such that $a_0h_0+b_0$ are prime numbers. Moreover, we know that if the independent term H_0 of an element $H \in \mathbb{Z}[\![x]\!]$ is prime in $\mathbb{Z}$, then H is prime in $\mathbb{Z}[\![x]\!]$ (see, for example, [BG08, Proposition 3.3.]). So, simply choose infinitely many $h \in \mathbb{Z}[\![x]\!]$ with independent term in the former collection of h_0's, and we obtain essentially infinitely many different prime elements of the form $ah + b$.

(ii) Assume that $\gcd(a_0, b_0) = d \neq 1$ is not a prime power. Then, for all $h_0 \in \mathbb{Z}$, $a_0h_0 + b_0 = d(a_0'h_0' + b_0')$ (with $\gcd(a_0', b_0') = 1$) is never a prime power, or it is zero. The case of being zero is only possible for $a_0' = a_0/d = 1$, and $h_0 = -b_0$. So, $ah + b = xs(x)$, for some $s(x) \in S$. Then, for all possible

values of $h_i \in \mathbb{Z}$, for $i \geqslant 1$, $ah+b$ is either reducible or an associate of the prime element $x \in S$. Thus, for this special subcollection of h's in S, $ah+b$ is essentially at most one irreducible element. For the rest of possible values of $h \in S$, the elements $a_0h_0 + b_0 \neq \pm 1$ are neither prime powers nor invertible elements. So, in these cases the corresponding power series are always reducible (see [BG08, Proposition 3.4.]). In conclusion, the version of Dirichlet's theorem does not hold in this case.

(iii) Assume that $\gcd(a_0, b_0) = p^m$, for some prime number p and some natural number $m > 0$ such that $p^{m+1}|a_0$ and $p^{m+1} \nmid b_0$. Then, for any $h \in \mathbb{Z}[\![x]\!]$, $a_0h_0 + b_0 = p^m(pa_0'h_0 + b_0')$, with $a_0' = a_0/p^{m+1} \in \mathbb{Z}$, and $\gcd(b_0', p) = 1$. So, $a_0h_0 + b_0$ is never zero, invertible, or a power of a prime since $pa_0'h_0 + b_0'$ is never zero, or divisible by p. In conclusion, again by [BG08, Proposition 3.4.] $ah+b$ is always reducible.

(iv) Firstly, the example given in Theorem 3.3 gives a case satisfying the hypothesis with essentially finite (indeed none) number of primes of the form $ah+b$ varying $h \in S$. Additionally, for the second possible conclusion, set $a = 6, b = 3 + x$, and $c = ah + b$ for $h \in S$. Note that a and b are coprime due to the fact that b is itself prime (since its independent term is prime) and it is different from all the prime factors of 6 which are 3 and 2 as power series. Moreover, note that $c_0 = 3(2h_0 + 1)$. Choose, for $i \in \mathbb{N}_{>0}$, $h_0^{(i)} \in \mathbb{Z}$ such that $2h_0^{(i)} + 1 = 3^i$, $h_1^{(i)} = 1$, and choose $h_j^{(i)}$ arbitrarily in $\mathbb{Z}$, for $j \in \mathbb{N}_{\geqslant 2}$. Then, $c_0^{(i)} = 3.3^i = 3^{i+1}$ and $c_1^{(i)} = a_0h_1^{(i)} + a_1h_0^{(i)} + b_1 = 7$. So, by [BGW12, Proposition 2.1.(c)] $c^{(i)}$ is prime for all $i \geqslant 1$. Moreover, if $i_1 \neq i_2$, then $c^{(i_1)}$ is not associated to $c^{(i_2)}$ because their independent terms are not associated in $\mathbb{Z}$ (remember that the units in S are exactly the series with an independent term ± 1 [BG08, Proposition 3.1]). In conclusion, we obtain essentially infinitely many irreducibles of the form $ah+b$ varying h in S. □

Remark 3.6. *The question of characterizing algebraically and arithmetically the pairs of series $(a, b) \in \mathbb{Z}[\![x]\!]^2$ for which there are essentially infinitely many primes in the series $ah + b$, varying h, in terms of suitable conditions for the coefficients a_i and b_i (with $i \in \mathbb{N}$), is a much more complex concern, partially, because, on the one hand, although there is an irreducibility criterion in $\mathbb{Z}[\![x]\!]$ (see* [Ell14, Theorem 1.4]*), it is not so easy to check the conditions given by Elliot in practice. On the other hand, there are different families of irreducible series sharing, for example, a fixed prime power independent term (see, for example, the collections of examples provided in* [BG08, §4]*).*

4. Goldbach-Dirichlet Representation for Some Polynomial Rings

The goal of this section is to present our main results about the Goldbach–Dirichlet representations introduced in Section 1 for some polynomial rings.

More precisely, our first main result is the following one.

Theorem 4.1. *Let $\mathbb{K}$ be a field of characteristic zero and let $S = \mathbb{K}[\mathbf{x}] := \mathbb{K}[x_1, \ldots, x_m]$, where $m \geqslant 1$. Then, S has SGD-representations of length $n+1$, for each $n \in \mathbb{N}$. In fact, it has infinitely many SGD-representations with arbitrarily large degree with respect to any fixed variable x_i.*

Proof. Let $Q \in S$, and $a_0, a_1, \ldots, a_n \in S$, $b_0, b_1, \ldots, b_n \in S$ fulfilling the conditions of Definition 1.4. Define the polynomials $P_i(y) := a_i y + b_i \in S[y]$, for $i = 1, \ldots, n$, and $P_0(y) := (\sum_{j=1}^{n} a_j)y + (\sum_{j=1}^{n} b_j) - Q \in S[y]$. Because of the assumptions that we impose along Definition 1.4 we can guarantee, as we have already pointed out along Remark 1.5, that $P_0, \ldots, P_n$ are irreducible linear polynomials in $S[y]$.

In conclusion, applying Theorem 1.2 to $P_0, P_1, \ldots, P_n \in S[y]$, we deduce that, in particular, there exists a polynomial $M(\mathbf{x}) \in S$ such that $Q_0 := P_0(\mathbf{x}, M(\mathbf{x})), \ldots, Q_n := P_n(\mathbf{x}, M(\mathbf{x}))$ are all primes in S. Now, set

$$M_0(\mathbf{x}) := \frac{-Q_0 - b_0}{a_0},\ R_0(\mathbf{x}) := a_0 M_0 + b_0,\ R_i(\mathbf{x}) := Q_i(\mathbf{x}),\ 1 \leqslant i \leqslant n.$$

By what we have already saw, all the R_i are irreducible polynomials of S. Moreover, by construction we also have that

$$Q = \sum_{i=0}^{n} R_i.$$

The last part of our theorem follows from the fact that due to Theorem 1.2 we can choose $M(\mathbf{x})$ of arbitrarily large degree with respect to any fixed variable x_i. This finishes our proof. □

Remark 4.2. *A very natural question at this stage is related with the existence of arbitrary GD-representations for integers (i.e., $S = \mathbb{Z}$) exactly in the manner that we stated Definition 1.4.*

For instance, if $n = 1$, $Q = 3$, $a_0 = b_0 = 1$ and $a_1 = 100$ and $b_1 = 1$ are parameters satisfying the conditions of Definition 1.4, then 3 cannot be written as the sum (or difference) of prime numbers $p_0 = a_0 x_0 + b_0$ and $p_1 = a_1 x_1 + b_1$, since each integer i that can be expressed as a sum of two primes of this form must fulfill the following conditions: either $i = 2$, or $i > 100$, or $i < -97$. In fact, the same argument applies for each integer $Q \in \{3, 4, 5, \ldots, 100\}$.

Even more, in the case of SGD-representations, we can also find counterexamples. Effectively, set $n = 1$, $Q = 20k + 1$ (for any $k \in \mathbb{Z}$), $a_0 = b_0 = b_1 = 1$ and $a_1 = 4$. So, the former parameters satisfy the conditions of Definition 1.4, since $\gcd(5, 2 - (20k + 1)) = 1$, due to the fact that 5 does not divide $2 - (20k + 1) = -20k - 1$. Furthermore, since the only possible residues of prime numbers modulo 4 are either $1, 2$ or 3 and the only primes allowed with Dirichlet's form $4m+1$ possess obviously remainder 1 modulo 4, then the sum for two primes with the former Dirichlet's form have residue either $0, 2$ or 3 modulo 4. Then, they never can express a number of the form $Q = 20k + 1$, which has residue 1 modulo 4.

Another way to look at this couple of examples is the following one. In the second case, we want to solve the system of equations and congruences

$$Y_0 + Y_1 = 20k + 1,\ Y_1 \equiv 1 \pmod 4.$$

The congruence is equivalent to $Y_1 = 1 + 4\lambda$ *for some* $\lambda \in \mathbb{Z}$*. In this way, it follows that* $Y_0 = 20k + 1 - Y_1 = 20k - 4\lambda = 4(5k - \lambda)$*, which can never take a prime value for any* $(k, \lambda) \in \mathbb{Z}^2$*.*

A similar argument works in the first case; indeed, in that case we want to solve the system $Y_0 = 3 - Y_1,\ Y_1 \equiv 1 \pmod{100}$*. This implies* $Y_0 = 2(1 - 50\lambda)$ *for some* $\lambda \in \mathbb{Z}$*, so* Y_0 *can never reach a prime value unless* $\lambda = 0$*.*

Remark 4.3. *Condition* (1.2) *can be phrased in terms of a linear system of equations. Indeed, under the assumptions of Definition 1.4 the existence of an SGD representation is equivalent to the following system of equations:*

$$\begin{cases} Y_0 + \ldots + Y_n = Q \\ Y_0 \equiv_\infty b_0 (\textit{mod}\ a_0) \\ Y_1 \equiv_\infty b_1 (\textit{modun}\ a_1) \\ \vdots \\ Y_n \equiv_\infty b_n (\textit{modun}\ a_n) \\ Y_0, \ldots, Y_n \in \mathbb{P}_S, \end{cases}$$

where $\mathbb{P}_S$ *denotes the set of irreducible elements of* S*, and "modun" means uniform modularity, in other words, that the elements given solutions to each of the congruences can be chosen uniformly as one, i.e., there is a kind of global solution of the system of congruences. Finally, the sub-index* ∞ *means that there are infinitely many uniform global solutions to the systems of equations. Note that in the proof of Theorem 4.1 the solution to the first congruence for* $i = 0$ *is not uniform (or global) regarding the global solution of the remainder ones. So, we do no write "modun" for it.*

Actually, a slightly stronger statement can be proved, we plan to use in a crucial way the following result, the interested reader can consult [BDN20, Theorem 1.3].

Theorem 4.4. *Assume that* $n \geqslant 2$ *is an integer, and that* $\mathbb{K}$ *is an arbitrary field. Let* $(a_1, b_1), \ldots, (a_s, b_s)$ *be* s *pairs of non–zero relatively prime polynomials in* $\mathbb{K}[x_1, \ldots, x_n]$*. Then, there is an integer* $d_0 \geqslant 1$ *satisfying the following property.*

For all integers $d_1, \ldots, d_n$ *larger than* d_0*, there exists an irreducible polynomial* $M \in \mathbb{K}[x_1, \ldots, x_n]$ *such that* $a_i + b_i M$ *is irreducible in* $\mathbb{K}[x_1, \ldots, x_n]$ *for any* $1 \leqslant i \leqslant s$*, and* $\deg_{x_j}(M) = d_j$ *for any* $1 \leqslant j \leqslant s$*.*

Replacing along the proof of Theorem 4.1 the use of Theorem 1.2 by Theorem 4.4 we obtain the following statement, which makes no assumptions on our base field $\mathbb{K}$.

Theorem 4.5. *Assume that $m \geqslant 2$ is an integer, and that $\mathbb{K}$ is an arbitrary field. Then, $S := \mathbb{K}[x_1, \ldots, x_m]$ has SGD-representations of length $n+1$, for each $n \in \mathbb{N}$. In fact, it has infinitely many SGD-representations with arbitrarily large degree with respect to any fixed variable x_i.*

Remark 4.6. *Note that it seems not straightforward how to generalize the proof of Theorem 4.1 for the case of GD-representations due to the fact that, if a_0 does not divides $\sum_{i=1}^{n} a_i$, then one cannot give so easily the form of $a_0 P_0'(\mathbf{x}) + b_0$ to the term $P_0(\mathbf{x})$.*

5. An Extension of the Green-Tao Theorem for Polynomial Rings in Characteristic Zero

The Green-Tao theorem over the integers establishes that there are arbitrarily large consecutive blocks of arithmetic progressions consisting entirely of prime numbers. In other words, for all $k \in \mathbb{N}$ there exist (coprime) integers a and b such that all the numbers $a+b, 2a+b, \ldots, ka+b$ are prime numbers [GT08, Theorem 1.1].

If we understand the concept of walking along polynomials in the standard sense of increasing the (total) degree and considering whole families of polynomials with (total) degree smaller than a fixed natural bound. For the sake of completeness we state explicitly the result given in [GOS23, Theorem 1].

Theorem 5.1. *Let D be an integral domain, let $n \geqslant 2$, let $k \geqslant 1$ be an integer, and set $A_k := \{p \in D[x_1, \ldots, x_n] : \deg(P) \leqslant k\}$. Then, setting*

$$f(x_1, \ldots, x_n) = x_n^{k+3} + x_1 \cdots x_{n-1}, \; g(x_1, \ldots, x_n) = x_1^2,$$

we have that, for any $h \in A_k$, $f + gh$ is irreducible in $D[x_1, \ldots, x_n]$.

Here, we establish a version of the Green-Tao theorem in the setting of a ring S equal either to the ring of polynomials in at least two variables over a field of characteristic zero, or to the ring of polynomials in at least one variable over the integers. In this setting, we can give a rough generalization of the notion of 'consecutive' element in the following sense. Of course, we can say that a polynomial $f(x)$ and $f(x)+1$ are consecutive since their difference is 1, which turns out to be another (very peculiar) polynomial (i.e. the multiplicative unit of the ring). Nonetheless, if we use, for a moment, the classic analogy establishing that the affine space $\mathbb{A}_k^1$ corresponds (or can be partially identified) with the ring of coordinates $k[x]$ via the well-known correspondence of points (given rise to directions or vectors) going to maximal (principal and non-zero in dimension one) ideals; then, we can understand a polynomial to be a kind of specific direction (or polynomial direction). In this intuitive correspondence one advances in the direction of $p(x)$ by considering integer positive multiples of the form $2p(x), 3p(x), 4p(x), \ldots$ and one goes

backwards with respect to the direction $p(x)$ if one considers negative multiples of the form $-p(x), -2p(x), -3p(x), \ldots$ In this way, the classic arithmetic notion of being consecutive to an element corresponds to the trivial polynomial direction $p(x) = 1$. Having this in mind let us establish formally our next theorem.

Theorem 5.2 (Extension of the Green-Tao Theorem for Polynomial Rings in Characteristic Zero)**.** *Let S be either $\mathbb{Z}[x_1, \ldots, x_n]$, with $n \geqslant 1$ or $\mathbb{K}[x_1, \ldots, x_m]$, where $\mathbb{K}$ is a field of characteristic zero and $m \geqslant 2$. Then, in S there exist arbitrarily long arithmetic progressions $\{b(\mathbf{x})(-) + a(\mathbf{x})\}$ going in any (vectorial) direction consisting entirely of primes (i.e. irreducibles). Even more, one can choose the polynomial $b(\mathbf{x})$ of arbitrarily large degree on any variable x_i.*

Proof. First, let us choose a fixed vectorial direction $p(\mathbf{x}) \in S\backslash\{0\}$, and a length $l \in \mathbb{N}$. Let us fix an arbitrary polynomial $a(\mathbf{x}) \in S$ being coprime with all the polynomials $p(\mathbf{x}), 2p(\mathbf{x}), \ldots, kp(\mathbf{x})$ (this is always possible because S is an UFD with infinitely many primes). Now, set $w = l$ in Theorem 1.2. So, by the former choice of $a(\mathbf{x})$ we have, keeping in mind Remark 1.1, that the polynomials $P_k(\mathbf{x}) := a(\mathbf{x}) + k \cdot p(\mathbf{x})y$ are all irreducible in $S[y]$ for $k = 1, \ldots, w$. Thus, by Theorem 1.2 there exists a polynomial $M = b(\mathbf{x})$ of arbitrarily large degree in any fixed variable x_j such that all the polynomials

$$a(\mathbf{x}) + p(\mathbf{x})b(\mathbf{x}), a(\mathbf{x}) + 2 \cdot p(\mathbf{x})b(\mathbf{x}), \ldots, a(\mathbf{x}) + w \cdot p(\mathbf{x})b(\mathbf{x})$$

are irreducible (i.e. prime) in S. But this means exactly that the arithmetic polynomial progression of length w

$$a(\mathbf{x}) + p(\mathbf{x})b(\mathbf{x}), a(\mathbf{x}) + 2 \cdot p(\mathbf{x})b(\mathbf{x}), \ldots, a(\mathbf{x}) + w \cdot p(\mathbf{x})b(\mathbf{x})$$

consists entirely of prime elements, finishing our proof. □

An even stronger result is true in this context with coefficients in a field, namely, that the former theorem remains true when we allow to perform not only 'regular' steps of multiples of the vectorial direction, but also we allow to make 'irregular' steps with an additional coprime vectorial variation at each (amplified) stage.

Theorem 5.3 (Polynomial Extension of the Green-Tao Theorem in Characteristic Zero with coprime irregular steps)**.** *Let $S = \mathbb{K}[x_1, \ldots, x_m]$, where $\mathbb{K}$ is a field of characteristic zero and $m \geqslant 2$. Then, for any vectorial direction $p(\mathbf{x}) \in S\backslash\{0\}$, and any arbitrarily large sequence of irregular coprime fluctuations of the vectorial steps given by a sequence of polynomials $a_1(\mathbf{x}), \ldots, a_l(\mathbf{x}) \in S$ all being coprime with vectorial direction $p(\mathbf{x})$, there exists a sort of irregular arithmetic progression $\mathcal{A}_{ip}$ of the form $b(\mathbf{x})(-) + a_1(\mathbf{x}), \ldots, b(\mathbf{x})(-) + a_l(\mathbf{x})$; for a suitable $b(\mathbf{x}) \in S$, such that the irregular path in the direction of $p(\mathbf{x})$ (multiplied by $b(\mathbf{x})$, i.e., amplified by $b(\mathbf{x})$) along $\mathcal{A}_{ip}$ consists exactly of l primes (or prime steps). Even more, one can choose the polynomial $b(\mathbf{x})$ of arbitrarily large degree on any variable x_i.*

Proof. As before, set $w = l$ as in Theorem 1.2. So, by Remark 1.1 the polynomials $P_i(\mathbf{x}) := a_i(\mathbf{x}) + i \cdot p(\mathbf{x})y$ are all irreducible in $S[y]$ for $i = 1, \ldots, w$, since our field has characteristic zero and $p(\mathbf{x})$ is coprime with all the $a_i(\mathbf{x})$. Then, by Theorem 1.2 there exists a polynomial $M = b(\mathbf{x})$ of arbitrarily large degree in any fixed variable x_j such that all the polynomials $a_1(\mathbf{x}) + p(\mathbf{x})b(\mathbf{x}), a_2(\mathbf{x}) + 2 \cdot p(\mathbf{x})b(\mathbf{x}), \ldots, a_w(\mathbf{x}) + w \cdot p(\mathbf{x})b(\mathbf{x})$ are irreducible (i.e. prime) in S. In other words, the arithmetic polynomial progression of length w

$$a_1(\mathbf{x}) + b(\mathbf{x})[p(\mathbf{x})], a_2(\mathbf{x}) + b(\mathbf{x})[2 \cdot p(\mathbf{x})], \ldots, a_w(\mathbf{x}) + b(\mathbf{x})[w \cdot p(\mathbf{x})],$$

which is the irregular path in the direction of $p(\mathbf{x})$ (amplified by multiplication with $b(\mathbf{x})$) along $\mathcal{A}_{ip}$, consists exactly of $w = l$ primes. □

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CONFLICT OF INTEREST

There is no conflict of interest.

REFERENCES

[BDN20] A. Bodin, P. Dèbes, and S. Najib. The Schinzel hypothesis for polynomials. *Trans. Amer. Math. Soc.*, 373(12):8339–8364, 2020. 3, 4, 13

[BG08] D. Birmajer and J. B. Gil. Arithmetic in the ring of formal power series with integer coefficients. *Amer. Math. Monthly*, 115(6):541–549, 2008. 8, 9, 10, 11

[BGR] A. F. Boix and D. A. J. Gómez-Ramírez. On some algebraic and geometric extensions of Goldbach's conjecture. Available at https://arxiv.org/pdf/2312.16524v1. 1, 2

[BGW12] D. Birmajer, J. B. Gil, and M. Weiner. Factoring polynomials in the ring of formal power series over $\mathbb{Z}$. *Int. J. Number Theory*, 8(7):1763–1776, 2012. 8, 11

[Cox22] D. A. Cox. *Primes of the form $x^2 + ny^2$—Fermat, class field theory, and complex multiplication*. AMS Chelsea Publishing, Providence, RI, third edition, 2022. With contributions by Roger Lipsett. 6

[Ell14] J. Elliott. Factoring formal power series over principal ideal domains. *Trans. Amer. Math. Soc.*, 366(8):3997–4019, 2014. 11

[FJ23] M. D. Fried and M. Jarden. *Field arithmetic*, volume 11 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer, Cham, fourth edition, 2023. 7

[Fos73] R. M. Fossum. *The divisor class group of a Krull domain*. Band 74, [Results in Mathematics and Related Areas]. Springer-Verlag, New York-Heidelberg, 1973. 4, 5, 8

[GOS23] H. Göral, H. B. Özcan, and D. C. Sertbaş. The Green-Tao theorem and the infinitude of primes in domains. *Amer. Math. Monthly*, 130(2):114–125, 2023. 3, 14

[GR20] D. A. J. Gómez Ramírez. *Artificial mathematical intelligence—cognitive, (meta)mathematical, physical and philosophical foundations*. Springer, Cham, 2020. 1, 2

[GT08] B. Green and T. Tao. The primes contain arbitrarily long arithmetic progressions. *Ann. of Math. (2)*, 167(2):481–547, 2008. 3, 14

[Jan73] G. J. Janusz. *Algebraic number fields*, volume 55 of *Pure and Applied Mathematics*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1973. 7

[Kha22] S. K. Khanduja. *A textbook of algebraic number theory*, volume 135 of *La Matematica per il 3+2, Unitext*. Springer, Singapore, 2022. 2, 7, 10

[LE23] A. Y. H. Lok Edison. *Euclidean proof of Dirichlet's theorem on arithmetic progressions in number fields*. PhD thesis, University of Warwick, 2023. 7

[Neu99] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. 6, 7

[Par20] E. Paran. Twin-prime and Goldbach theorems for $\mathbb{Z}[\![x]\!]$. *J. Number Theory*, 213:453–461, 2020. 3

[Pin09] J. Pintz. Landau's problems on primes. *J. Théor. Nombres Bordeaux*, 21(2):357–404, 2009. 2

[Sam61] P. Samuel. On unique factorization domains. *Illinois J. Math.*, 5:1–17, 1961. 8

[SS58] A. Schinzel and W. Sierpiński. Sur certaines hypothèses concernant les nombres premiers. *Acta Arith.*, 4:185–208, 1958. 2

VISIÓN REAL COGNITIVA (COGNIVISIÓN) S.A.S. ITAGUÍ, COLOMBIA.
Email address: daj.gomezramirez@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH, AV. EDUARD MARISTANY 16, 08019, BARCELONA, SPAIN.
Email address: alberto.fernandez.boix@upc.edu