Homological Conjectures, Closure Operations, Vector Bundles and Forcing Algebras

Tesis de doctorado

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Introduction

There are two main sources for producing and doing research in mathematics:

First, by finding and analyzing good examples, from which we get the main intuitions for establishing and proving conjectures. On this approach the examples turn out to be as important as theorems, and sometimes they are the "down-to-earth" versions of the theorems, and they clarify, as well, the need for imposing technical conditions to the general propositions. Sometimes good examples are as valuable as good theorems. In fact, they can have a big influence in a whole theory.

A good instance of that is the counterexample to the localization problem in commutative algebra given by H. Brenner and P. Monsky (see [**6**]), which settled a fundamental question in one of the most important theories of commutative algebra: *Tight Closure* (see [**25**]). Since the invention of the theory in the late 80s, the question of determining if this particular closure operation commutes with localization was on the basis of the subsequent research. However, neither the creators nor the purely commutative algebrists came with the solution to this problem. In fact, the main intuitions for finding the counterexample came from geometry. Specifically, from the work of H. Brenner relating vector bundles and torsors (see [**3**]). His work is rich of examples and geometrical and homological intuitions of the algebraic phenomena.

In this particular case, an specific example turned out to be as important as a whole abstract theory. In fact, after the acceptance of the counterexample in the mathematical community the horizon in tight closure theory has changed. Particularly, the new directions are going to create new theories to go further. For example, to create theories of closure operations that "commute with localization" (see [**12**]).

Secondly, there is a way of looking for the general reasons of the mathematical phenomena. Instead of searching for particular examples we consider at the same time different kinds of abstract theories as specific instances to work with, and we try to find common properties among them in order to create new even more abstract theories which have the former theories just as particular examples.

One important instance is the modern abstract algebraic geometry, where the very fundamental definition to consider i.e., the notion of scheme, involves intuitions and notions coming from differential topology, as well as from the theory of sheaves, and from commutative algebra. One of the brightest exponent of this way of thinking is Alexander Grothendieck, whose main philosophy of solving a mathematical problem could be explained as creating a whole and suitable mathematical environment for the problem (i.e. an appropriate theory) in such a way that the solution of the problem follows naturally from the adequate definitions and formal constructions. A classical example of that is the Grothendieck-Riemann-Roch Theorem (see [**11**]). In comparison with the former way this goes on the other direction, since instead of looking for an specific example to get intuition about the problem in consideration i.e., going from the micro-evidence to reach the general patterns describing the desired solution, we look for a suitable global instance in order to derive our particular problem just as a natural and specific example of a "big" formal construction.

These two ways of doing mathematical research have been very successful, although the modern mathematic tendency goes more on the second direction.

In this thesis we go, in a very modest way, through these two possible forms of doing mathematics.

First, we work with a very natural object in commutative algebra: the forcing algebras. We consider a commutative ring with unity R, a finitely generated $I = (f_1, ..., f_n) \subseteq R$ and another arbitrary element $f \in R$. An important question, not only from the theoretical, but also from the computational point of view is to determine if f belongs or not to I (or, in general. to some closure operation on I, such as the radical, or the tight closure, among others). Sometimes, we try to determine in a well defined way how "close" f is from belonging to I. One possible formal way to approach this question is by finding another suitable $R-$algebra A such that f belongs to the corresponding expansion of I in this algebra i.e., $f \in IA$. Specifically, we need to know if there exist elements $t_1, ..., t_n \in IA$, such that $f = f_1t_1 + \cdots + f_nt_n$. A simple and elegant possibility to find such an algebra is by formally "forcing" the former equation i.e., by defining the desired coefficients just as formal variables T_i, and dividing the ring of polynomials over R on these variables by the "forcing" equation $f_1T_1 + \cdots + f_nT_n - f$. Specifically, we consider the "forcing algebra"

$$A = R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n - f)\,,$$

with data $f, f_1, ..., f_n \in R$. Originally the notion of forcing algebra appeared in the work of M. Hochster on *Solid Closure* (see [**23**]). It turns out that we can translate the fact that f belongs to I or to some closure operation of I in terms of verifying specific algebraic as well as topological properties of the corresponding forcing algebra and the corresponding "forcing morphism"

$\varphi : Y := \operatorname{Spec} A \to X := \operatorname{Spec} R$, induced on the corresponding affine schemes by the natural homomorphism $i : R \to A$. We describe in very general terms the most important examples of this correspondence regarding the most important closure operations in commutative algebra (Ch. 1, §2).

Although forcing algebras involve the most elementary and simple system of equations, i.e. linear equations, they have lots of interesting nontrivial properties. However, they have been seldom studied as an subject on their own. On the other hand, studying forcing algebras is, in a general sense, studying linear algebra, but not only over an arbitrary commutative ring, but also carrying the topological, geometrical and homological structure of the involved system of linear equations, which is, clearly an interesting topic on its own. So, we do this in a modest and specific way in the first three chapters of this thesis.

Particularly, we study the case corresponding to a submodule N of a finitely generated module M and an arbitrary element $s \in M$. This case corresponds to forcing algebras with several forcing equations

$$A = R[T_1, \ldots, T_n] / \left\langle \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} + \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \right\rangle .$$

Even very basic properties of forcing algebras are not yet understood, and these thesis deals to a large extension with these questions.

For example, we describe how to perform elementary row and column operations on the forcing algebra by means of considering elementary affine linear isomorphisms and an specific relation between the regular sequences of forcing elements and the fitting ideals of the corresponding forcing matrix (Ch. 1, §3).

Further, the connectedness of the spectrum Y of the forcing algebra can be essentially described by means of the "vertical" and "horizontal" components regarding the forcing morphism φ. And, in lower dimensions, the role of the horizontal component becomes even more important (Ch 2, §2-3). Furthermore, if our base ring is a unique factorization domain (UFD), then a purely arithmetical condition is sufficient to guarantee the connectedness of the forcing algebra, and on the principal ideal domain it becomes necessary too. Moreover, if we move to Dedekind domains then, under mild conditions, the connectedness of Y is translated exactly as the belonging of f to I (Ch 2, §4).

On the other hand, regarding connectedness as a local property over the base X, a quite general result holds: for any arbitrary homomorphism of rings $\alpha : R \to A$, to guarantee the connectedness of $\operatorname{Spec} A$, it is sufficient to prove the local (over the base) connectedness of it i.e., to verify that $\operatorname{Spec} A_{R \setminus \mathfrak{p}}$ is connected for any $\mathfrak{p} \in \operatorname{Spec} R$. However, we show with an example coming from forcing algebras that the converse of this fact does not hold. However, the local connectedness is equivalent to the global one in the case of forcing algebras over a noetherian one dimensional domain (Ch 2, §5). Being in the integral closure could be completely characterized by the universal connectedness of the forcing morphism, and, in particular, when a fraction is integral over a noetherian domain (Ch. 2, §6). All of these previous results are explained not only with formal proofs but also with simple and rich examples.

The irreducibility of the forcing algebra over a noetherian domain can be obtained just by assuming that the height of I is bigger or equal that 2 (Ch 3, §1).

Now, we show with two kinds of examples that for the reducedness of the forcing algebras is not enough to have the reducedness of the base. Besides, as a natural consequence of studying this and doing very elementary considerations we see that a noetherian ring is the product of fields if and only if any element belongs to the ideal generated by its square power (Ch. 3, §1). Moreover if we add to the condition the possibility that I is the whole base R, then we get a complete characterization of the integrity of the forcing algebra over UFDs (Ch. 3, §3).

With a very natural approximation through simple examples and increasing just step by step the dimension of the base space we obtain, in the case that our base is the ring of polynomials over a perfect field, a quite simple criterion of normality for the forcing algebras by means of the size of the codimensions of the ideal I and the ideal $I + D$, where D is generated by the partial derivatives of the data. In the case that we are working over an algebraic closed field and our base is the ring of coordinates of an irreducible variety X, the normality of the (forcing) hyperplane defined by the forcing equation can be characterized by imposing the condition that the codimension of the singular locus of X in the whole affine space is a least three (Ch 3, §4). Here, it is worth to know that we present the formal proof of this criterion as well as the "informal" way in which this criterion was originally found i.e., a way of analyzing simple examples increasing gradually the generality of the variables describing them, in order to develop slowly a deeper intuition of the phenomenon involved. We suggest by working on this criterion and going for a moment to the philosophy of mathematics, the possibility to combine the two former approaches in a common way consisting of creating and developing a "theory of examples" in mathematics. Where we consider examples as our formal objects, and we try to formalize the notion of rich examples and how they can "converge" to other more general examples or theorems, and lastly, how we can visualize each particular collection of topics or theories in mathematics just as specific and suitable examples to work with.

As an instance of the importance of the examples we analyze an specific forcing algebra, that we call the "enlightening" example, because it is a very natural recurring point to verify the different results that we have already studied. On this respect this example is not less important that the former results. Instead of that, it is another valuable result where the different propositions and theorems come together (Ch. 3, §5).

Moreover, we compute explicitly the normalization of a forcing algebra coming from the examples guiding us to find the normality criterion. And again, on that process we deal with very elementary and fundamental questions dealing with normal domains and denominators ideals (Ch. 3 §6).

In the second part of this thesis we study one of the most important homological conjecture: The Direct Summand Conjecture (DSC), which states that if $R \hookrightarrow S$ is a finite extension of rings and R is regular then R is a direct summand of S as $R-$module, or, equivalently, this extension splits as a map of $R-$modules i.e. there exists a retraction or $R-$homomorphism $\rho : S \to R$ sending 1_R to 1_S. We discuss very briefly the state of the art of this conjecture and its equivalent version, the Monomial Conjecture stating that if (R, m) is a local ring of dimension d and $\{x_1, ..., x_n\}$ a system of parameters, then for any $t \in \mathbb{N}$ we have

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, ..., x_d^{t+1}).$$

Clearly, this form of the DSC goes into the direction of looking for a counterexample, (Ch.1 §4).

On the other hand, J. D. Vélez, in his former work has reformulated the DSC conjecture in terms of the existence of annihilators of zero divisors on Gorenstein local rings not belonging to ideals of parameters (Ch. 4, §3-4). Based on these results we find a new conjecture equivalent to the DSC (in its weak form) (Ch. 4 §6). It states in its strong form that if (T, η) is a Gorenstein local ring of dimension d and $\{x_1, ..., x_d\} \subseteq T$ is a system of parameters and writing $Q = (x_1, ..., x_d)$, then for any zero divisor $z \in T$ and any lifting $u \in T$ of a socle element in T/Q (i.e. $\mathrm{Ann}_{T/Q}(\bar{\eta}) = (\bar{u})$) then

$$u \cdot z \in Q \cdot (z).$$

This (a little bit technical) condition allows for much more flexibility to make computations on particular examples than the original statement of the DSC. We call this conjecture the Socle-Parameters Conjecture (Strong Form) (SPCS). The SPCS is at the same time equivalent to a very general and homological condition involving the lengths of the Koszul homology groups i.e., to saying that

$$\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0.$$

Note that the condition involved is, in some respects, more suitable to be generalized in a brighter mathematical context beyond commutative algebra, because it essentially involves homological estimates, which at the same time can be rewritten by means of the Euler characteristic or by the multiplicity of the ring $T/(z)$. Besides, the main ring in question is a Gorenstein ring which is basically defined by an homological property i.e., having finite injective dimension. In virtue of that we get a new and brighter horizon over the DSC, which allows us to find a new proof of the DSC in the positive equicharacteristic case (Ch. 4, §7). Besides, both conditions together with a theorem of Ikeda (see [**26**, Corollary 1.4]), helps us to show the SPCS for multiplicities of T smaller or equal than two, suggesting a natural induction as a way of solving the conjecture (Ch. 4, §8). In chapter 4 we argue more with general abstract arguments than with specific examples, so we go more into the second way of doing mathematics.

Finally, we turn again to the elementary examples and prove the DSC for finite extensions S of UFDs generated by two elements satisfying quadratic radical equations, and, on the way, we prove an interesting and more general fact characterizing when rings of the form

$$R[T_1, T_2](f_1(T_1), f_2(T_2)),$$

are integral domains, where R is also a domain and f_1, f_2 are monic polynomials (Ch. 5, §1). Besides, by imposing a couple of arithmetical conditions on the coefficients and on the discriminants of the polynomials who have as root the generators of S as $R-$module we proof also the DSC for non-radical quadratic extensions (Ch. 5, §2).

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CHAPTER 1

Fundamental Definitions and Results

1. Basic Definitions and Results

A *local* ring R (or (R, m, k)) is a Noetherian ring with a unique maximal ideal m and residue class field $k = R/m$. The *characteristic* of R is the minimal positive integer r such that $r \cdot 1 = 0$, that is, the positive generator of the kernel of the natural homomorphism from $\mathbb{Z}$ to R sending 1 to 1. If this homomorphism is injective, we define the characteristic of R as zero. We say that R is *equicharacteristic* if $\mathrm{char} R = \mathrm{char} k = p$, with $p \geq 0$. It is an elementary fact that R contains a field if and only if R is equicharacteristic. We say that R is *mixed characteristic* if $\mathrm{char} k = p$ and $\mathrm{char} R = p^{\alpha}$ ($\alpha \neq 0$), here $p > 0$ because if $\mathrm{char} k = 0$. Then $\mathrm{char} R = 0$. It is elementary to see that the characteristic of a local ring can only be zero or a power of the characteristic of its residue field.

As usual k (or K) shall denote an arbitrary field, except it is explicitly otherwise defined.

If M is an $R-$module, then an element $x \in R$ is a zerodivisor on M if there exists a nonzero element $a \in M$ such that $xa = 0$. A sequence $x_1, ..., x_n \in R$ is a *regular sequence* for M (or an $M - sequence$) if x_i is a nonzero divisor in $M/(x_1, ..., x_{i-1})M$ for $i = 1, ..., n$ and $(x_1, ..., x_n)M \neq M$. For R Noetherian, M finitely generated $R-$module and $I \subseteq R$ and ideal such that $IM \subsetneq M$, the length of a maximal regular sequence on M contained in I is a well defined integer called the *depth* of M in I or the $I - depth$ and denoted by $\mathrm{depth}(I, M)$ (see [**30**, Theorem 16.7]). If $IM = M$. Then, by definition, $\mathrm{depth}(I, M) = +\infty$.

An $R-$module M is *projective* if the functor $\mathrm{Hom}_R(M, -)$ is exact (i.e. preserves short exact sequences). A sequence

$$... \to P_i \xrightarrow{d_i} P_{i-1} \to ... \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

is a *projective resolution* of M, if each P_i is a projective $R-$module and $\ker(d_i) = \mathrm{im}(d_{i-1})$. If there exists some $N \in \mathbb{N}$ such that $P_i \neq 0$ for all $i = 0, ..., n$ and $P_{n+1} = 0$, then n is called the length of this resolution. Otherwise, we say that this length is infinite. The *projective dimension* of M, denoted by $\mathrm{pd}_R M$, is defined as the infimum of the lengths of projective resolutions of M. In the case that there is no such projective resolution we write $\text{pd}_R M = +\infty$. Let (R, m) be a local ring an M be a finitely generated $R-$module with $\text{pd}_R M < +\infty$. Then the Auslander-Buchsbaum formula states that $\text{pd}_R M = \text{depth}(m, R) - \text{depth}(m, M)$ (see [**10**, Theorem 19.9]).

The Krull *dimension* of a ring S, denoted by $\dim S$, is the supremum of n such that there exists $P_0 \subsetneq P_1 \subsetneq ... \subsetneq P_n$, where each P_i is a prime ideal of S. Besides, the *height* of a prime ideal P of S, or *codimension* of P at S is defined by $\text{codim}(P, S) = \text{ht}P := \dim R_P$. The height, or codimension, of an arbitrary ideal I, is defined as the infimum of the heights of the minimal prime ideals of I. We will denote this number by $\text{codim}(I, S)$, emphasizing that our ideal is contained in the ring S. One can also define the notion of codimension of closed subset within a topological space (see [**17**, p. 86]), coinciding with the former one in the case that R is identified with the affine scheme $\text{Spec}\, R$, consisting of the prime ideals of R with Zariski topology, i.e., the closed subset are of the form $V(J)$, for some ideal $J \subseteq R$, and consist of the prime ideals containing J. In conclusion it holds $\text{codim}(V(I), \text{Spec}\, R) = \text{codim}(I, R)$ (for further readings see [**17**]).

If M is an $R-$module then $\dim M := \dim(R/\text{Ann}_R M)$. Another important fact that we use is that $\dim(R[x_1, ..., x_n]) = \dim R + n$, for any Noetherian ring (see [**30**, Theorem 15.4]).

A ring R is catenary if any saturated chain of primes between any two prime ideals P and Q, with $P \subseteq Q$, have the same length. And R is universally catenary if any finitely generated $R-$algebra is catenary. The dimensions of quotients of catenary domains behave quite natural. Specifically, $\dim R/I = \dim R - \text{ht}I$ for any ideal $I \subseteq R$ (see [**10**, p. 290]). The most common example of a universally catenary ring is an affine domain, i.e. a finitely generated $k-$algebra, which is an integral domain (see [**10**, Corollary 13.6.]). In fact, let I be an ideal of an affine domain $T = k[x_1, \ldots, x_r]/P$, where P is a prime ideal of $k[x_1, \ldots, x_r]$ and k is an arbitrary field, then

$$dim(T) = tr.deg_k K(T) = dim(T/I) - ht(I).$$

Moreover, $dim(T)$ is the length of every maximal chains of primes in T (see [**10**, Theorem A, p. 290]). We use this result particularly oft in Chapter 3.

If (R, m, k) is a local ring and M is a finitely generated $R-$module, then M is called *Cohen-Macaulay* (C-M) if $\text{depth}(I, M) = \dim M$. In the case that R is itself a C-M $R-$module, we say that R is a C-M ring. Globally, the C-M notion is defined if the C-M condition holds in every localization on prime ideals. Moreover, a local ring R is C-M if and only if one (therefore any) *system of parameters* $x_1, ..., x_d \in R$ (i.e. $\text{Rad}(x_1, ..., x_d) = m$, and $\dim R = d$) is a regular sequence on R (see[**30**, Theorem 17.4.]). Finally, any local C-M ring (R, m) is *equidimensional*, (i.e. the heights of all maximal ideals are the same and for any minimal prime $P \subseteq R$, the dimensions of R/P are the same) and the associated primes are minimal (see [**10**, Corollary 18.10, Corollary 18.11]).

A ring R is called *Dedekind domain* if is a Noetherian domain of dimension one such that any localization on nonzero prime ideals are discrete valuation rings, i.e. local principal ideal domains.

Let (R, m) be a local ring, one form of the well known Lemma of Nakayama is the following: Let $N \subseteq M$ be $R-$modules such that M/N is finitely generated and $M = N + mM$, then $M = N$ (see [**30**, Corollary p.8]).

A theorem of Krull says that on a Noetherian ring a prime ideal P has height n if and only if it is minimal over an ideal generated by n elements (see [**10**, Theorem B, p. 224]). A local ring R is *regular* if the minimal number of generators of the maximal ideal, $\mu(m) = \dim_k(m/m^2)$, (by Lemma of Nakayama) is exactly the Krull dimension of R. In general, by Krull's theorem $\mu(m) \geq \dim R$. Any regular local ring is a UFD, and then, in particular, a normal domain, i.e. coincides with its integral closure on its field of fractions (see [**10**]). Besides, regular local rings are C-M (see [**10**]). A noetherian ring is regular is its localizations at prime ideals are regular local rings. Finally, if R is a regular ring, then $R[x_1, ..., x_d]$ so is (see [**30**, Theorem 19.5])

A local ring R is *Gorenstein* if it is C-M and for one (therefore any) system of parameters $\{x_1, ..., x_n\} \subseteq R$ the dimension of the socle $\mathrm{Ann}_{R/(x_1,...,y_n)}(\overline{m})$ is one as a $k-$vector space. A generator of the socle is called a socle element. For equivalent definitions (see [**30**, Theorem 18.1.]).

Let M be an $R-$module. Let J be a directed system and $\{M_j\}_{j \in J}$ a family of submodules of M such that if $s \geq r$, then $M_s \subseteq M_r$. The *completion* of M with respect to this family is defined by $\widehat{M} := \varprojlim M/M_j$. If $J = \mathbb{N}$ and $M_j = I^j M$, where $I \subseteq R$ is an ideal, then $\widehat{M}$ is the completion of M with respect to I (or $I-$adic completion). R is *complete* with respect to I, if the natural homomorphism from R to the $I-$adic completion of R sending r to $(\overline{r})$ is an isomorphism. A local ring (R, m, k) is complete if it is $m-$adic complete. For example, if $m^n = 0$ for some $n \in \mathbb{N}$, then R is complete. In general, taking completion preserve (sometimes in both directions) a lot of useful properties of R such that the Krull dimension, depth, regularity, and being C-M, among others. For further reading see [**30**] and [**10**].

A local ring (R, m) is *unmixed* if its completion $\widehat{R}$ is equidimensional (see [**29**, p. 247]).

An $R-$module M is *flat* if $- \otimes M$ preserves short exact sequences, which is equivalent to saying that it preserves injectivity, since it is elementary to see that tensoring with an arbitrary module is a right exact functor. Let M, N be $R-$modules. The functor $\mathrm{Tor}_n^R(N, M)$ is defined by fixing any projective resolution of M or N, tensoring with the other module and taking homology. This module is well defined up to isomorphism. We will need the following two facts: if one of the factors is flat as an $R-$module, then $\mathrm{Tor}_n^R(N, M) = 0$ for $n > 0$. The other fact is that for any short exact sequence

$$0 \to A \to B \to C \to 0,$$

there exists a long exact sequence

$$\cdots \to \mathrm{Tor}^R_{i+1}(C,M) \xrightarrow{\delta_i} \mathrm{Tor}^R_i(A,M) \to \mathrm{Tor}^R_i(B,M) \to \mathrm{Tor}^R_i(C,M) \xrightarrow{\delta_{i-1}} \cdots$$

$$\mathrm{Tor}^R_1(C,M) \xrightarrow{\delta_0} A\otimes M \to B\otimes M \to C\otimes M \to 0$$

(see [**30**, Appendix B, p. 278]).

An $R-$module N is *faithful* if $N \otimes_R M = 0$, implies $M = 0$ for any $R-$module M. If $R \hookrightarrow \widehat{R}$ is the natural extension of a local ring (R, m, k) to its completion then, it is a faithfully flat extension (see [**30**, p. 63]).

An extension of rings $R \hookrightarrow S$ is *finite* (or *module-finite*) if S is a finitely generated $R-$module. If $R \hookrightarrow S$ is an integral extension of rings then by the going up (see [**10**, Proposition 4.15]) $\dim S = \dim R$. Since finite extensions are integral, the same holds in that case. In general, a ring homomorphism $h : R \to S$ is *finite*, if $h(R) \hookrightarrow S$ is a finite extension. We say that h *splits* if there exists a $R-$module homomorphism $\rho : S \to R$ such that $\rho \circ h = id_R$, or what is the same, ρ is $R-$linear (S consider as an $R-$module via h) and $\rho(1) = 1$.

A generalization of the "Nullstellensatz" (see [**10**]) tells us that the ring of polynomials over an arbitrary field is a *Jacobson* ring. That means, it is a ring such that any prime ideal is an intersection of maximal ideal, or equivalently, any prime ideal is exactly the intersection of all the maximal ideals containing it.

A ring R is Artinian if any descending chain of ideals stabilizes, which is equivalent to saying that R is a Noetherian ring of dimension zero (see [**1**, Theorem 8.5.]). It is also true that Artinian rings have finitely many maximal ideal, saying $\{m_1, ..., m_r\}$. In this case R is naturally isomorphic to $R_{m_1} \times ... \times R_{m_r}$ (see [**1**, Theorem 8.7.]).

Let R be a commutative ring with unity and $\{x_1, ..., x_n\} \subseteq R$ a sequence of elements of R. If $p < 0$, let $K_p = 0$. If $p = 1, ..., n$, define

$$K_p = \oplus_i Re_{i_1,..,i_p},$$

where $i = (i_1, ..., i_p)$ and $1 \le i_1 < i_2 < ... < i_p \le n$; and finally $K_0 = R$. Let $d_p : K_p \to K_{p-1}$ defined by

$$e_{i_1,...,i_p} \to \sum_{j=1}^{p} (-1)^j e_{i_1,...,\widehat{i_j},...,i_p}.$$

Now, it is elementary to see that $K_\bullet$ define a complex of $R-$modules, i.e., $d_p \circ d_{p-1} = 0$. The Koszul homology is defined by

$$H_r(\underline{x}, R) := h_r(k_\bullet) = \ker(d_r)/\mathrm{im}(d_{r+1}).$$

If M is an $R-$module then $H_r(\underline{x}, M) := h_r(M \otimes K_\bullet)$. One of the most useful properties of the Koszul homology is the following fact: Let (R, m) be a local, $I = (x_1, ..., x_n) \subseteq R$ an ideal, M a finitely generated $R-$module and define

$\mu = \sup\{r : H_r(\underline{x}, M) \neq 0\}$. Then, $\text{depth}(I, M) = n - \mu$ (see [**30**, Theorem 16.8]).

A composition series for an $R-$module M is a sequence $M_0 = (0) \subsetneq M_1 \subsetneq ... \subsetneq M_\ell = M$, such that M_i/M_{i-1} is a simple $R-$module. If a composition series exists, then any two of them have the same length, which is called the length of M and it is denoted by $\ell(M)$. Otherwise, we define $\ell(M) = +\infty$. Moreover, $\ell(M) < +\infty$ if and only if M satisfies both the ascending and descending chain conditions. In particular, Artinian rings have finite length as modules over themselves. Finally, $\ell(-)$ is additive on short exact sequences. For further reading see [**30**, Chapter 6].

Let R be an $\mathbb{N}-$graded ring such that R_0 is an Artinian ring and such that R is finitely generated as an R_0-algebra. Let M be a finitely generated $R-$module of dimension d. Then it is elementary to see that each homogeneous part M_n is a finitely generated R_0-module and therefore has finite length (see [**1**, Proposition 6.5.]). Besides, there exists a unique polynomial

$$p_M(t) = a_{d-1}t^{d-1} + \cdots + a_0 \in \mathbb{Q}[t]$$

of degree $d-1$, which is called the *Hilbert polynomial*, such that for $n \gg 0$, $p_M(n) = \ell(M_n)$. The *multiplicity* of M, $e(M)$, is defined as $\ell(M)$ if $d = 0$, and as $(d-1)!a_{d-1}$, if $d > 0$ (see [**7**, Definition 4.1.5.]. In particular, if M has positive dimension, then $e(M) > 0$, since $a_{d-1} > 0$. For the local case, assume that (R, m) is a local ring, M a finitely generated $R-$module of dimension d and $I = (x_1, ..., x_n)$ is an *ideal of definition* of M. This last condition means that $m^r M \subseteq IM$ for some $r > 0$, which is equivalent to saying that $x = x_1, \cdots, x_n$ is a *multiplicity system* on I (i.e. $\ell(M/(x_1, ..., x_n)M) < +\infty$) (see [**7**, p. 185]). We define the filtered graded ring $\text{gr}_I R = \oplus_{i=0}^{+\infty} I^i/I^{i+1}$, and the filtered graded module $\text{gr}_I M = \oplus_{i=0}^{+\infty} I^i M/I^{i+1}M$, where $I_0 = R$. Then, $\text{gr}_I R$ is in a natural way a graded ring (here R/I is Artinian, because after reducing to the case $\text{Ann}_R M = 0$, it is easy to see that $\ell(M/IM) < +\infty$ if and only if $\text{rad} I = m$). Therefore we can define the multiplicity of M on I, $e(I, M) := e(\text{gr}_I M)$ (see [**7**, p. 180]) and the multiplicity of R, $e(R) := e(m, R)$ (see [**30**, p. 108]). In particular, we can define the multiplicity of M on $I = (x_1, ..., x_n)$, where $x_1, ..., x_n \in R$ is a *system of parameters* of M, i.e. $n = \dim M$ and M is Artinian, i.e. satisfied the descending chain condition for submodules (see [**1**, p. 74]). Besides, under the former hypothesis and assuming that $x = x_1, \cdots, x_n$ is a multiplicity system of M, we can define the *Euler Characteristic* as

$$\chi(\underline{x}, M) := \sum_{i=0}^{n} (-1)^i \ell(H_i(\underline{x}, M)).$$

For a more technical reformulation of this notion due to Auslander and Buchsbaum, see [**7**]. Now, a theorem of Serre (see [**7**, Theorem 4.6.6.]) states that $\chi(\underline{x}, M) = e(I, M)$, if $\{x_1, ..., x_N\} \subseteq R$ is a system of parameters for M

and zero otherwise. In particular, if $\{x_1, ..., x_n\} \subseteq R$ is a system of parameters for M and $\dim M > 0$, then $\chi(\underline{x}, M) = e(I, M) = e(\mathrm{gr}_I M) > 0$, because $\dim M = \dim(\mathrm{gr}_I M)$ (see [**7**, Theorem 4.4.6.]).

Now, we review the notion of a coefficient ring: If (R, m, k) is equicharacteristic, a *coefficient field* is a field $K_0 \subseteq R$ such that the natural projection $\pi : K_0 \subseteq R \hookrightarrow R/m = k$ is an isomorphism. On the other hand, let (R, m, k) be a complete quasi-local (that means with a unique maximal ideal m but not necessarily Noetherian), mixed characteristic, and *separated* ring (i.e. $\cap_{n\in\mathbb{N}} m^n = (0)$). Then a *coefficient ring* for R is a sub-ring $(D, \eta) \hookrightarrow R$ such that it is complete, local, $\dim D \leq 1$, $m \cap D = \eta$, and the inclusion induces an isomorphism $D/\eta \cong R/m$. It is elementary to see that if $\mathrm{char} R = 0$, and $\mathrm{char} k = p > 0$, then D is a domain, and therefore one dimensional, that means exactly that (D, η) is a discrete valuation domain (DVD). A theorem of I. S. Cohen states that for complete local rings, there always exists a coefficient ring (see [**8**, Theorem 9, Theorem 11] and [**19**, p. 24]). In fact, in the mixed characteristic case ($\mathrm{char} k = p > 0$) there exists coefficient rings which are DVD-s (D, pD, k), with local parameter p, or $D/p^m D$, when $\mathrm{char} R = p^m$ (see [**19**, Corollary p. 24]).

Let
$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$$
be a short exact sequence of $R-$modules. It is elementary to prove the equivalence of the following facts:

(1) A is a direct summand of B via this exact sequence. That means that there exists an $R-$isomorphism $\theta : B \to A \oplus C$ inducing an isomorphism of short exact sequence with the canonical sequence $0 \to A \to A \oplus C \to C \to 0$, with the identity on A and C.
(2) There is a retraction (or splitting $R-$homomorphism) $\rho : B \to A$, i.e. $\rho \circ i = \mathrm{Id}_A$.
(3) There exists an $R-$homomorphism $\alpha : C \to B$ such that $\alpha \circ \pi = Id_C$.
(4) The induced $R-$homomorphism $\theta : \mathrm{Hom}_R(C, B) \to \mathrm{Hom}_R(C, C)$ given by $\beta \to \beta \circ \pi$ is surjective.

In particular, if $R \hookrightarrow S$ is a ring extension and the previous conditions hold for the naturally induced short exact sequence $0 \to R \to S \to S/R \to 0$, then we say that this extension *splits*. Note that if $R \hookrightarrow R'$ is a faithfully flat extension, then $R \hookrightarrow S$ splits if and only if $R' \hookrightarrow R' \otimes_R S$ splits. In fact, since for flat extensions tensor products commutes with the functor Hom (see [**2**, §2, No. 10, Proposition 11]), it is elementary to see that $\mathrm{Hom}_R(S/R, S) \to \mathrm{Hom}_R(S/R, S/R)$ is surjective if and only if $\mathrm{Hom}_{R'}((R' \otimes S)/R', R' \otimes S) \to \mathrm{Hom}_{R'}((R' \otimes S)/R', (R' \otimes S)/R')$ is surjective, which means that $R' \hookrightarrow R' \otimes S$ splits.

Remark 1.1. Let R be a commutative ring and $B = R[T_1, \ldots, T_n]$ the polynomial ring in n variables. Let P be a prime ideal of R, then for the expansion of P in B, say PB, holds $\text{ht}PB \geq \text{ht}P$. In fact, if $P_0 \subsetneqq P_1 \subsetneqq, \ldots, \subsetneqq P_r = P$ is a saturated chain of primes in R contained in P, then $P_0B \subsetneqq P_1B \subsetneqq, \ldots, \subsetneqq P_rB = PB$ is a saturated chain of primes in B, since the quotients $B/P_iB \approxeq (R/P_i)[T_1, \ldots, T_n]$ are all different integral domains, due to the fact that the functor $- \otimes_R R[T_1, \ldots, T_n]$ is faithfully flat (since $R[T_1, \ldots, T_n]$ is a free $R-$module) and tensoring the exact sequence of $R-$modules

$$0 \longrightarrow P_i/P_{i-1} \longrightarrow R/P_{i-1} \longrightarrow R/P_i \longrightarrow 0$$

we obtain the exact sequence:

$$0 \longrightarrow P_iB/P_{i-1}B \longrightarrow B/P_{i-1}B \longrightarrow B/P_iB \longrightarrow 0,$$

so $P_iB/P_{i-1}B \neq 0$.

We state explicitly the statement of the Jacobian Criterion and a corollary of it, which we use in the proof of our normality criterion in Chapter 3, for proofs see [**10**, Theorem 16.19, Corollary 16.20].

THEOREM 1.2. *(Jacobian Criterion). Let $S = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k, let $I = (f_1, \ldots, f_s)$ be an ideal, and set $R = S/I$. Let P be a prime ideal of S containing I and write $k(p) = K(R/P)$ for the residue class field at P. Let c the codimension of I_p in S_p.*

1. *The Jacobian matrix $J = (\partial f_i/\partial x_j)$, taken modulo P, has rank $\leq c$.*
2. *If $\text{char}K = p > 0$, assume that $k(P)$ is separable over k. R_P is a regular local ring if and only if the matrix J, taken modulo P, has rank $= p$.*

COROLLARY 1.3. *Let $R[x_1, \ldots, x_r]/I$ be an affine ring over a perfect field k and suppose that I has pure codimension c, i.e., the height of any minimal prime over I is exactly c. Suppose that $I = (f_1, \ldots, f_n)$. If J is the ideal of R generated by the $c \times c$ minors of the Jacobian matrix $(\partial f_i/\partial x_j)$, then J defines the singular locus of R in the sense that a prime P of R contains J if and only if R_P is not a regular local ring.*

Now we present the statement of Serre's Criterion for normality for any Noetherian ring (see [**10**, Theorem 11.2.]). Let us recall that a ring is normal if it is the direct product of normal domains:

THEOREM 1.4. *A Noetherian ring S is normal if and only if the following two conditions holds:*

1. *(S2) For any prime ideal P of S holds*
$$\text{depth}_P(S_P) \geq \min(2, \dim(S_P)).$$
2. *(R1) Every localization of S on primes of codimension at most one is a regular ring.*

Sometimes we will use the phrase "previous comments" or "previous results", to refer, among other things, the results stated here.

2. Forcing algebras

Let R be a commutative ring, $I = (f_1, \ldots, f_n)$ a finitely generated ideal and f an arbitrary element of R. As mentioned in the introduction, a very natural and important question, not only from the theoretical but also from the computational point of view, is to determine if f belongs to the ideal I or to some ideal closure of it (for example to the radical, the integral closure, the plus closure, the solid closure, the tight closure, among others). To answer this question the concept of a forcing algebra introduced by Mel Hochster in the context of solid closure [**23**] is important (for more information on forcing algebras see [**4**], [**5**]):

DEFINITION 1. Let R be a commutative ring, $I = (f_1, \ldots, f_n)$ an ideal and $f \in R$ another element. Then the *forcing algebra* of these (forcing) data is

$$A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)\,.$$

Intuitively, when we divide by the forcing equation $f_1T_1 + \ldots + f_nT_n + f$ we are "forcing" the element f to belong to the expansion of I in A. Besides, it has the universal property that for any R-algebra S such that $f \in IS$, there exists a (non-unique) homomorphism of R-algebras $\theta : A \to S$.

Furthermore, the formation of forcing algebras commutes with arbitrary change of base. Formally, if $\alpha : R \to S$ is a homomorphism of rings, then

$$S \otimes_R A \cong S[T_1, ..., T_n]/(\alpha_1(f_1)T_1 + \cdots + \alpha_n(f_n)T_n + \alpha(f))$$

is the forcing algebra for the forcing data $\alpha(f_1), \ldots, \alpha(f_n), \alpha(f)$. In particular, if $\mathfrak{p} \in X = \operatorname{Spec} R$, then the fiber of (the forcing morphism) $\varphi : Y := \operatorname{Spec} A \to X := \operatorname{Spec} R$ over $\mathfrak{p}$, $\varphi^{-1}(\mathfrak{p})$, is the scheme theoretical fiber $\operatorname{Spec}(\kappa(\mathfrak{p}) \otimes_R A)$, where $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is its residue field. In this case, the fiber ring $\kappa(\mathfrak{p}) \otimes_R A$ is the forcing algebra over $\kappa(\mathfrak{p})$ corresponding to the forcing data $f_1(\mathfrak{p}), \ldots, f_n(\mathfrak{p}), f(\mathfrak{p})$, where we denote by $g(\mathfrak{p}) \in \kappa(\mathfrak{p})$, the image (the evaluation) of $g \in R$ inside the residue field $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Also, note that for any f_i $A_{f_i} \cong R_{f_i}[T_1, ..., \check{T}_i, ..., T_n]$, via the $R_{f_i}-$homomorphism sending $T_i \mapsto -\sum_{j \neq i}(f_j/f_i)T_j - (f/f_i)$ and $T_r \mapsto T_r$ for $r \neq i$.

An extreme case occurs when the forcing data consists only of f. Then, we define I as the zero ideal. Therefore $A = R/(f)$.

Besides, if $n = 1$, then intuitively the forcing algebra $A = R[T_1]/(f_1T_1 - f)$ can be consider as the graphic of the "rational" function f/f_1. We will explore this example in more detail in chapter two.

By means of forcing algebras and forcing morphisms one can rewrite the fact that the element f belongs to a particular closure operations of I. We shall illustrate this now.

Firstly, the fact that $f \in I$ is equivalent to the existence of a homomorphism of $R-$algebras $\alpha : A \to R$, which is equivalent at the same time to the existence of a section $s : X \to Y$, i.e. $\varphi \circ s = Id_X$.

Secondly, f belongs to the radical of I if and only if φ is surjective. In fact, suppose that φ is surjective and let us fix a prime ideal $\mathfrak{p} \in X$ containing I. Then, $\varphi^{-1}(x) = \operatorname{Spec} \kappa(\mathfrak{p}) \otimes A \neq \emptyset$, that means, $\kappa(\mathfrak{p}) \otimes A = \kappa(\mathfrak{p})[T_1, ..., T_n]/(f_1(\mathfrak{p})T_1 + \cdots + f_n(\mathfrak{p})T_n + f(\mathfrak{p})) \neq 0$. But, each $f_i(\mathfrak{p}) = 0$, since $f_i \in \mathfrak{p}$, therefore $f(\mathfrak{p})$ is also zero, thus $f \in \mathfrak{p}$. In conclusion, $f \in \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \operatorname{rad} I$. Conversely, suppose that $f \in \operatorname{rad} I$ and take an arbitrary prime $\mathfrak{p} \in X$. Then, if I is not contained in $\mathfrak{p}$, then some $f_j(\mathfrak{p}) \neq 0$ and so $\kappa(\mathfrak{p}) \otimes A \neq 0$, that means $\varphi^{-1}(\mathfrak{p}) \neq \emptyset$. Lastly, if $I \subseteq \mathfrak{p}$ then $f \in \mathfrak{p}$, and therefore $\kappa(\mathfrak{p}) \otimes A = \kappa(\mathfrak{p})[T_1, ..., T_n] \neq 0$ and thus $\varphi^{-1}(\mathfrak{p}) = \mathbb{A}^{n-1}_{\kappa(\mathfrak{p})} \neq \emptyset$. In conclusion φ is surjective.

Thirdly, let us review the definition of the tight closure of an ideal I of a commutative ring R of characteristic $p > 0$. We say that $u \in R$ belongs to the *tight closure* of I, denoted by I^*, if there exists a $c \in R$ not in any minimal prime, such that for all $q = p^e \gg 0$, $cu^q \in I^{[q]}$, where $I^{[q]}$ denotes the expansion of I under the $e-$th iterated composition of the Frobenius homomorphism $F : R \to R$, sending $x \to x^p$. Tight Closure is one of the most important closure operations in commutative algebra and was introduced in the 80s by M. Hochster and C. Huneke as an attempt to prove the "Homological Conjectures" (for more information [**25**]). Let (R, m) be normal local domain of dimension two. Suppose that $I = (f_1, ..., f_n)$ is an $m-$primary ideal and f is an arbitrary element of R. Then, $f \in I^*$ if and only if $D(IA) = \operatorname{Spec} A \setminus V(IA)$ is not an affine scheme, i.e. is not of the form Spec D for any commutative ring D (see [**5**, corollary 5.4.]).

Forth, the origin of the forcing algebras comes from the definition of the solid closure, as an effort to defining a closure operation for any commutative ring, independent the characteristic (see [**23**]). Explicitly, let R be a Noetherian ring, let $I \subseteq R$ an ideal and $f \in R$. Then, f belongs to the *solid closure* of I if for any maximal ideal m of R and any minimal ideal $\mathfrak{q}$ of its completion $\widehat{R}_m$, for the complete local domain $(R' = \widehat{R}_m/\mathfrak{q}, m')$ holds that the $d-$th local cohomology of the forcing algebra A', obtained after the change of base $R \hookrightarrow R'$, $H^d_m(A') \neq 0$, where $d = \dim R'$ (see [**3**, Definition 2.4., p. 15]).

Fifth, let us consider an integral domain R and an ideal $I \subseteq R$. Then, u belongs to the *plus closure* of I, denoted by I^+, if there exists a finite extension of domains $R \hookrightarrow S$, such that $f \in IS$. If R is a Noetherian domain and $I = (f_1, ...f_n) \subseteq R$ is an ideal and $f \in R$, then $f \in I^+$ if and only if there exists an irreducible closed subscheme $\widetilde{Y} \subseteq Y = \operatorname{Spec} A$ such that $\dim \widetilde{Y} = \dim X$, $\varphi(\widetilde{Y}) = X$ and for each $x \in X$, $\varphi^{-1}(x) \cap \widetilde{Y}$ is finite (for an projective version of this criterion see [**3**, Proposition 3.12]).

Finally, if R denotes an arbitrary commutative ring and $I \subseteq R$ is an ideal, then we say that u belongs to the *integral closure* of I, denote by $\overline{I}$, if there exist $n \in \mathbb{N}$, and $a_i \in I^i$, for $i = 1, ..., n$, with

$$u^n + a_1 u^{n-1} + \cdots + a_n = 0.$$

We will prove in Chapter 2, §6 that $f \in \overline{I}$, where $I = (f_1, ..., f_n) \subseteq R$, if and only if the corresponding forcing morphism φ is universally connected, i.e. $\operatorname{Spec}(S \otimes_R A)$ is a connected space for any Noetherian change of base $R \to S$, such that $\operatorname{Spec} S$ is connected.

From this we derive a criterion of integrity for fractions $r/s \in K(R)$, where R denotes a Noetherian domain, in terms of the universal connectedness of the natural forcing algebra $A := R[T]/(sT + r)$.

In view of this results, it seems very natural to study in commutative algebra the question of finding a closure operation with "good" properties (see [**12**]), in terms of finding suitables algebraic-geometrical as well as topological or homological properties of the forcing morphism. This approach goes closer to the philosophy of Grothendieck's EGA of defining and studying the objects in a relative context (see [**17**] and [**15**]). A simple and deep example of this approach is the counterexample to one of the most basic and important open questions on tight closure: the Localization Problem i.e., the question whether tight closure commutes with localization. This was done by H. Brenner and P. Monsky using vector bundles techniques and geometric deformations of tight closure (see [**3**]).

Besides, another good example going in this direction is a general definition of forcing morphism for arbitrary schemes. Specifically, let X and Y be arbitrary schemes. Suppose that $i : Z \to X$ is a closed subscheme and $f \in \Gamma(X, \mathcal{O}_X)$ is a global section. Then, a morphism $\varphi : Y \to X$ is a *forcing morphism* for f and Z, if

i) the pull-back of the restriction of f to Z, $f_{|Z} = i^{\sharp}_Z(f)$ is zero, i.e. $\varphi^{\sharp}_{|\varphi^{-1}(Z)}(f_{|Z}) = 0$;

ii) for any morphism of schemes $\psi : W \to X$ with the same property, i.e. $\psi^{\sharp}_{|\psi^{-1}(Z)}(f_{|Z}) = 0$, there exists a (non-unique) morphism $\widetilde{\psi} : W \to Y$ such that $\psi = \varphi \circ \widetilde{\psi}$. It is a natural generalization of the universal property of a forcing algebra but in the relative context and in a category including that of commutative rings with unity.

3. Forcing Algebras with several Forcing Equations

Now, we study just a few elementary properties of forcing algebras which are defined by several forcing equations and which leads us in a natural way to the understanding of the linear algebra over the base ring R. This section could be understood as a simple invitation to this barely explored field of mathematics. Here we recommend for further reading [**4**]. In this case we can write the forcing algebra in a matrix form:

$$A = R[T_1, \dots, T_n]/ \left\langle \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \dots & f_{mn} \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} + \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \right\rangle .$$

This corresponds to a submodule $N \subseteq M$ of finitely generated R-modules and an element $f \in M$ via a free representation of these data (see [**4**, p. 3]).

Now, we study how the forcing algebra behaves when we make elementary row or column operations in the associated matrix M. Remember that the matrix notation in the forcing algebra just means that we are considering the ideal generated by the rows of the resulting matrix, after performing the matrix multiplications and additions.

First, if $l_1, ..., l_m$ denote the rows of M, and $c \in R$ denote an arbitrary constant, making a row operation, $l_j \mapsto cl_j + l_i$, $(i \neq j$; that is changing the jth row by c times the ith row plus the jth row) just means changing the generators $h_1, ..., h_m$ to the new generators $h_1, ..., h_{j-1}, ch_i + h_j, h_{j+1}, ..., h_m$. The ideal generated by these two groups of forcing elements coincides and therefore the associated forcing algebra are the same. Similarly, if we make operations of the form $l_i \mapsto l_j$ and $l_i \mapsto cl_i$, where c is an invertible element of R, which correspond to change two rows and to multiply a row by an element in R, then the forcing algebra does not change.

For the column operation, the problem is a little bit more subtle. Let $\{C_1, ..., C_n\}$ be the columns of the matrix A. Consider the column operation $\mapsto dC_i + C_j C_j$, where $d \in R$. Now, define the following automorphism φ of the ring of polynomials $R[T_1, ..., T_n]$ sending $T_s \mapsto T_s$, for $s \neq i$, and $T_i \mapsto cT_j + T_i$. Now,

$$\varphi(h_r) = f_{r1}T_1 + ... + f_{ri}(cT_j + T_i) + ... + f_{rn}T_n =$$

$$f_{r1}T_1 + ... + (cf_{ri} + f_{rj})T_j + ... + f_{rn}T_n.$$

and then φ induces an isomorphism between the forcing algebra with matrix M and the forcing algebra with matrix obtained from M performing the previous column operation. Similarly, for operations of the form $C_i \mapsto C_j$ and $C_i \mapsto dC_i$, where $d \in R$ is an invertible element, the resulting forcing algebras coincide. Now, if R is a field and the rank of the associated matrix M is r, where $r \leq \min(m, n)$, then performing row and column operations on the associated matrix we can obtained a matrix form by the $r \times r$ identity matrix in the upper-left side and with zeros elsewhere.

Therefore, the elements h_i have just the following simple form: $h_i = T_i + g_i$, for $i = 1, ..., r$ and $h_i = g_i$, for $i > r$, and some $g_i \in R$ (this g_i could appear just in the nonhomogeneous case, corresponding to the changes made on the independent vector form by the f_j). Thus the forcing algebra A is isomorphic either to zero (in the case that there exists $g_i \neq 0$, for some $i > r$)

or to $k[T_{r+1},...,T_n]$. This allow us to present the following lemma describing the fibers of a forcing algebra as affine spaces over the base residue field.

LEMMA 3.1. *Let R be a commutative ring and let A be the forcing algebra corresponding to the data $\{f_{ij}, f_i\}$. Let $\mathfrak{p} \in X$ be an arbitrary prime ideal of R and r the rank of the matrix $\{f_{ij}(\mathfrak{p})\}$. Then the fiber over $\mathfrak{p}$ is empty or isomorphic to the affine space $\mathbb{A}^{n-r}_{\kappa(\mathfrak{p})}$.*

PROOF. We know by a previous comment in section 1 that the fiber ring over $\mathfrak{p}$ is $\kappa(\mathfrak{p}) \otimes_R A$ which is just

$$\kappa(\mathfrak{p})[T_1,\ldots,T_n]/\left\langle \begin{pmatrix} f_{11}(\mathfrak{p}) & \ldots & f_{1n}(\mathfrak{p}) \\ \vdots & \ddots & \vdots \\ f_{m1}(\mathfrak{p}) & \ldots & f_{mn}(\mathfrak{p}) \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} + \begin{pmatrix} f_1(\mathfrak{p}) \\ \vdots \\ f_m(\mathfrak{p}) \end{pmatrix} \right\rangle .$$

Now, making elementary row and column operations on the matrix $(f_{ij}(\mathfrak{p}))$, as indicated before, we can obtain a matrix with zero entries except for the first r entries of the principal diagonal which are ones, plus an independent vector.

In conclusion, after performing all the necessarily elementary operations, we obtain an isomorphism from A to a very simple forcing algebra

$$B = \kappa(\mathfrak{p})[T_1,...,T_n]/(T_1 + g_1,...,T_r + g_r, g_{r+1},...,g_n),$$

corresponding to the matrix with zero entries except for the first r entries of the principal diagonal, which are ones. But then B is clearly isomorph to the affine ring $\kappa(\mathfrak{p})[T_{r+1},...,T_n]$, if $g_{r+1} = \cdots = g_n = 0$, and $A = 0$ otherwise, proving our lemma.

□

If $\kappa(\mathfrak{p})$ is algebraically closed, then the fiber over a point $\mathfrak{p} \in \operatorname{Spec} R$ of this forcing algebra is just the solution set of the corresponding system of inhomogeneous linear equations over $\kappa(\mathfrak{p})$. If the vector $(f_1,\ldots,f_m)$ is zero, then we are dealing with a "homogeneous" forcing algebra. In this case there is a (zero- or "horizontal") section $s : X = \operatorname{Spec} R \to Y = \operatorname{Spec} A$ coming from the homomorphism of R-algebras from A to R sending each T_i to zero. This section sends a prime ideal $\mathfrak{p} \in X$ to the prime ideal $(T_1,\ldots,T_n)+\mathfrak{p} \in Y$.

REMARK 3.2. If all f_k are zero, and $m = n$, then the ideal $\mathfrak{a}$ is defined by the linear forms $h_i = f_{i1}T_1 + \cdots + f_{in}T_n$, and in this case we can "translate" the fact of multiplying by the adjoint matrix of M, denoted by adj$M$, just to saying that the elements $\det MT_i \in \mathfrak{a}$. In fact,

$$\begin{pmatrix} \det MT_1 \\ \vdots \\ \det MT_n \end{pmatrix} = \det M \cdot I_{nn} \cdot \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} = \operatorname{adj} M \cdot M \cdot \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} = \operatorname{adj} M \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

where the entries of the last vector belong to $\mathfrak{a}$. From this fact we deduce that, when the determinant of M is a unit in R, then $\mathfrak{a} = (T_1, ..., T_n)$ and the forcing algebra is isomorphic to the base ring R. Note that the previous argument works also in the nonhomogeneous case.

Now, we study the homogeneous case when the elements $\{h_1, ..., h_m\}$ form a regular sequence. First, we need the following general fact about the pure codimension of regular sequences in Noetherian rings.

PROPOSITION 3.3. *Let S be a Noetherian ring and $\{r_1, ..., r_m\} \subseteq S$ a regular sequence and I the ideal generated by these elements. Then the pure codimension of I is m.*

PROOF. We make induction over m. For $m = 1$ we know that every minimal prime ideal of (r_1) has height ≤ 1 by Krull's theorem and the height cannot be zero because r_1 is not contained in any minimal prime of S, because it is not a zero divisor. Suppose that $I_{m-1} = (r_1, ..., r_{m-1})$ has pure codimension $m-1$ and write $S' = S/I_{m-1}$. Let P be a minimal prime over I on S. We know that r_m is not a zero divisor in S' and therefore for the case $m = 1$ the pure codimension of (r_m) in S' is one. Let's denote by P' the prime ideal corresponding to P in S'. Then P' is minimal over r_m and therefore its height in S' is one. Let P'_0 be a minimal prime on S' contained in P' and P_0 its corresponding prime in S. Then P_0 is minimal over I_{m-1} and thus by induction hypotheses it has height $m-1$ in S. Consider a saturated chain of primes of length $m-1$ ending in P_0

$$Q_0 \nsubseteq Q_1 \nsubseteq ... \nsubseteq Q_{m-1} = P_0.$$

Thus, completing this chain just by adding P we obtain a saturated chain of length m for P. This shows that $\mathrm{ht}P \geq m$, but by Krull's theorem we know that this height is $\leq m$, because P is minimal over an ideal generated by m elements on a Noetherian ring. This proves our proposition.

□

Besides, if $j \in \{1, ..., \min(m, n)\}$, then we define the Fitting ideals I_j as the ideals generated by the minors of size j of the matrix M. This definition corresponds to the standard definition of Fitting ideals regarding M as a R-homomorphism of free modules (see [**10**, p. 497]).

PROPOSITION 3.4. *Let R be a Noetherian integral domain and A the homogeneous forcing algebra corresponding to the data $\{f_{ij}\}$, with $i = 1, ..., n$ and $j = 1, ..., m$. Suppose that the forcing equations $\{h_1, ..., h_m\}$ form a regular sequence in $B := R[T_1, ..., T_n]$. Then $m \geq n$ and $I_{\min(m,n)} \neq (0)$.*

PROOF. First, note that the ideal I generated by the forcing elements is contained in the homogeneous ideal $P = (T_1, ..., T_n)$, therefore we see that the dimension of A is smaller or equal to the dimension of B/P, which is exactly the dimension of R. On the other hand, if we consider a saturated chain of primes in A,

$$P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_{\dim A},$$

where P_0 is a minimal prime over I, then by Proposition 3.3 $\mathrm{ht}(P_0) = m$ and thus completing the former chain with a saturated chain for P_0 in B of length m, we see that $\dim B \geq m + \dim A$. Now, noting that $\dim B = \dim A + n$, since A is Noetherian, we get $n + \dim A \geq m + \dim A$, which implies that $n \geq m$.

For the second part, let's consider the matrix M in the field of fractions K of R. It is an elementary fact that the rank of M is $\leq s$ if and only if every minor of size $s+1$ of M is zero. This follows from the fact that performing row operations on a matrix change the values of fixed minor of size r of the original matrix just by a nonzero constant term of another minor of size r of the changed matrix (this is a general way of saying that performing a row operation is just multiplying by an invertible matrix and therefore the fact that the determinant of the matrix is zero or not is independent of the row operation).

Now, suppose by contradiction that $I_{\min(m,n)} = 0$, then the rank of M in K is strictly smaller than $\min(m,n)$ and thus the rows of M are linearly dependent in K. Without loss of generality, assume that there is $j \in \{1, ..., \min(m,n)\}$, such that the jth row of M, l_j, is a linear combination of the former ones, that is, there exist some $\alpha_i \in K$, such that $l_j = \sum_{i=1}^{j-1} \alpha_i l_i$. Now, after multiplying by a nonzero common multiple $\beta \in R$ of the denominators, we get an equation of the form $\beta l_j = \sum_{i=1}^{j-1} \gamma_i l_i$, for some $\gamma_i \in R$. But, seeing this equality in A, (which means just multiplying this equality by the $n \times 1$ vector given by the T_i) we see that $\beta h_j = \sum_{i=1}^{j-1} \gamma_i h_i$, which implies that h_j is a zero divisor in $B/(h_1, ..., h_{j-1})$, because $\beta \notin (T_1, ..., T_n)$ (β is a nonzero constant polynomial in B) and therefore $\beta \notin (h_1, ..., h_{j-1})$. This contradicts the fact that I is generated by a regular sequence. □

The converse of the previous proposition is false as the following example shows.

Example 3.5. Consider $R = k[x]; B = R[T_1, T_2]; h_1 = xT_1 - xT_2$ and $h_2 = xT_1 + xT_2$, where k is a field of $\mathrm{char} k \neq 2$. Then $m = n = 2$ and the determinant of the associated matrix is $2x^2$, but the sequence $\{h_1, h_2\}$ is not regular, in fact, the ideal I generated by its elements has height just one, because it is contained in the principal ideal (x), and therefore by Proposition 3.3, I cannot be generated by a regular sequence. Geometrically, the variety defined by I is the union of a line $V(T_1, T_2)$ and a plane $V(x)$.

Intuitively, this example comes from the following observation. Suppose that we have the forcing algebra with equations $h'_1 = T_1 + T_2$ and $h_2 = T_1 - T_2$. If we consider the line $V(T_1 - T_2, T_1 + T_2) = V(T_1, T_2)$ (whose associated determinant is $2 \neq 0$) and multiplying these equations by x, we obtain a variety that is automatically the union of this line with the plane $V(x) \subseteq k^3$, which has bigger dimension, but the associated determinant of the new variety (our former example) is just x^2 times the former determinant. This process gives us a new variety with nonzero determinant but with an ideal with smaller codimension.

4. Some Homological Conjectures

The Homological Conjectures are a collection of conjectures on commutative algebra relating homological as well as structural properties of commutative rings, (involving its depth, dimension and projective dimension, among others). They appeared in the 70's, mainly after the work J. P. Serre on the theory of multiplicities (see [**36**]) and their ultimate version, as today appears, was stated by M. Hochster (see [**24**]). One of the most important ones is the Direct Summand Conjecture, implying all the standard homological conjectures in the positive characteristic case (see [**22**] other references are [**33**], [**35**], [**31**]). In this section we discuss a little bit the state of art of this conjecture, in order to introduce the reader to Chapter 4, where we give a new equivalent form of it and prove some instances.

Direct Summand Conjecture (DSC). Let $R \hookrightarrow S$ be a finite extension of rings, where R is a regular ring. Then $R \hookrightarrow S$ splits, as a map of $R-$modules.

It is proved in [**20**, Theorem 1] that the DSC holds if and only if for any system of parameters $x_1, ..., x_d \in R$ and any natural number t,

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, ..., x_d^{t+1})S.$$

Explicitly, this is the well-known Monomial Conjecture (MC):

Monomial Conjecture Let (R, m) be a local ring of dimension d and $\{x_1, ..., x_n\}$ a system of parameters. Then for any $t \in \mathbb{N}$ we have

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, ..., x_d^{t+1}).$$

The (MC) is equivalent to the Direct Summand Conjecture (DSC)(see [**21**] or [**22**, Theorem 6.1]).

By elementary methods we can reduce the DSC to the local regular complete case for R (see proof of Theorem 6.1 (Ch.4)). By the Going Up, we can assume that S is an integral domain.

Firstly, if R contains a field of characteristic zero, then we consider the trace between the corresponding field of fractions, and after considering its restriction to S and dividing by the degree of the extension we obtain the desired retraction (see [**20**, Lemma 2]).

Secondly, if R contains a field of positive characteristic, then, again by doing elementary considerations on the fields of fractions, we see that there exists a constant $c \in R$, such that cS is contained in a free $R-$module lying on the field of fractions of S. Besides, after reducing to the case in which R contains a perfect field and by means of considering iterations of the Frobenius homomorphism (i.e. the homomorphism elevating to the characteristic of R) we obtain the splitting homomorphism. We expose in section 6 of Chapter 4 a new proof of this case using estimates coming from the Theory of Multiplicities. Moreover, we can prove the conjecture in the equicharacteristic zero case by assuming positive equicharacteristic case for arbitrary large primes. The result is obtain be means of Model Theory, specifically by using Leftschetz's Principles, and Schautens' methods of the theory of ultrafilters, in logic (for further readings and similar results concerning Koh's Conjecture see [**38**], [**13**]).

Thirdly, for low dimensions the state of the DSC is the following:

If $\dim R \leq 2$, then after passing to the normalization of S, which is again a finite extension of S, (see [**27**, Exercise 9.8]), We can assume that S is a local normal domain of dimension smaller of equal than two and finitely generated $R-$module. So, we can apply Serre's Criterion for normality to see that S is a CM ring and thus the formula from Auslander-Buchsbaum shows that S has projective dimension zero, i.e., S is $R-$free. In that way we get the splitting. Lastly, we show, in Remark 2.1 (Ch.3), the case $d = 1$, by general and elementary considerations, via the (MC).

The case of dimension three was obtain by R. Heitmann and his proof is a significant advance in commutative algebra (see [**18**]).

The mixed characterictic case is open for dimension bigger that three.

Finally, an important reduction of the DSC was obtain by Hochster and we use it in Chapter 4 (see [**22**, Theorem 6.1. and proof]). Specifically, in order to prove the DSC we can assume that R is a unramified (i.e. $p \notin m^2$, where $p = \mathrm{char}(R/m)$) complete regular local of mixed characteristic ring with algebraically closed residue field k and S is an integral domain.

For the main role of the DSC in Commutative Algebra and further references, see [**24**].

CHAPTER 2

Connectedness

1. Generalities

Let us recall that a topological space X is connected if there exist exactly two subsets (namely $\emptyset$ and $X \neq \emptyset$) which are open as well as closed. A connected component of X is a maximal connected subspace, i.e., not strictly contained in any connected subspace of X. Every connected component is necessarily closed because its closure is a closed connected set containing it. Moreover, the connected components form a partition of the space X.

Let Z be a set and let F be a subset of the power set $P(Z)$ of parts of Z. We say that Z has the zig-zag property (zzp) for F if for any pair of subsets $A, B \in F$ there exist finitely many subsets $Y_1, ..., Y_m \in F$ such that $Y_1 = A$, $Y_m = B$ and for any $i = 1, ..., m-1$, holds that $Y_i \subseteq Y_{i+1}$ or $Y_{i+1} \subseteq Y_i$. Let X be a Noetherian topological space, that means, that in X holds the descending chain condition for closed subsets. Thus, every closed subset of it can be written uniquely as a finite union of irreducible closed subsets $X_1, ..., X_n$ (irreducible components) no one of them contained in the union of the other ones (see [**17**, Proposition I 1.5.].) Then, it is elementary to see that X is a connected topological space if and only if X has the zig-zag property for the set consisting of the irreducible components.

For a commutative ring A, the spectrum $\operatorname{Spec} A$ is connected if and only if $A \neq 0$ and if it is not possible to write $A = A_1 \times A_2$ with $A_1, A_2 \neq 0$. Equivalently, there exist exactly two idempotent elements, namely 0 and 1 (see for example [**17**, Exercise 2.19, Chapter II] or [**10**, Exercise 2.25]). Hence domains and local rings are connected.

If A is an algebra of finite type over $\mathbb{C}$, i.e. $A = \mathbb{C}[x_1, ..., x_n]/J$, for some ideal $J \subseteq \mathbb{C}[x_1, ..., x_n]$. Then, the connectedness of $Y := \operatorname{Spec} A$, with the Zariski topology, is equivalent to the connectedness of $Y_{\mathbb{C}} = \operatorname{Spm} A = V(J) \subseteq \mathbb{C}^n$ with the standard topology. We shall sketch a proof of this fact mainly following [**37**, Theorem 1, p. 126]. For another approach see [**32**, Lemma p. 56].

First, after considering a decomposition into the irreducible components and since they are, clearly, connected in the Zariski topology, we can assume that our variety is irreducible. We argue by induction on $d = \dim Y$. If $n = 1$, then after passing to the normalization we can assume that Y is nonsingular. But, on that case we derive the connectedness of $Y_{\mathbb{C}}$ as a direct consequence of the classification theorems for nonsingular complex curves. If $n > 1$, by Bertini's Theorem we derive the existence of a surjective morphism $\varphi : Y \supseteq U \to Z$, where U is open, Z is an irreducible variety of dimension $n-1$ and the fibers are irreducible curves (and therefore $\varphi_{\mathbb{C}}^{-1}(u)$ is connected for all $u \in U$). Besides, φ has surjective differential in every point of U. Now, assume that there exits closed subsets $Y_1, Y_2 \in Y_{\mathbb{C}}$ such that $Y_{\mathbb{C}} = Y_1 \uplus Y_2$, then by the assumptions on φ, $Z_{\mathbb{C}} = \varphi_{\mathbb{C}}(Y_1) \uplus \varphi_{\mathbb{C}}(Y_2)$, where by induction we know that $Z_{\mathbb{C}}$ is connected. Furthermore, by the surjectivity of the differentials and the Theorem of the Implicit Function $\varphi_{\mathbb{C}}(Y_1)$ and $\varphi_{\mathbb{C}}(Y_2)$ are open, therefore one of them is empty. In conclusion, one of the Y_i is empty, which implies the connectedness of $U_{\mathbb{C}}$. Finally, we see again by induction on d that $U_{\mathbb{C}}$ is dense on $Y_{\mathbb{C}}$, so $Y_{\mathbb{C}}$ is also connected.

Since an affine space is irreducible and hence connected, the preceding lemma tells us that the fibers of a forcing algebra are connected unless they are empty. The easiest example of an empty forcing algebra is $K[T]/(0T-1)$. A forcing algebra may be connected though some fibers may be empty, an example is given by $K[X,Y][T_1,T_2]/(XT_1+YT_2-1)$.

In the following we will mainly deal with the case where all fibers are non-empty. This is equivalent to say that f belongs to the radical of the ideal I (or, by definition, to the radical of the submodule N, see [**4**, Example 3.1.]).

PROPOSITION 1.1. *Let A be a forcing algebra over R with the corresponding morphism $\varphi : Y = \operatorname{Spec} A \to X = \operatorname{Spec} R$. Then the following hold.*

1. *The connected components of Y are of the form $\varphi^{-1}(Z)$ for suitable subsets $Z \subseteq X$.*
2. *If $\varphi : Y \to X$ is surjective, then these Z are uniquely determined.*
3. *If $\varphi : Y \to X$ is surjective and Y is connected, then X is connected.*
4. *If the forcing data are homogeneous, then there is a bijection between the connected components of X and Y. In particular, X is connected if and only if Y is connected.*
5. *Suppose that $\varphi : Y \to X$ is a submersion. Then there is a bijection between the connected components of X and Y. In particular, X is connected if and only if Y is connected.*

PROOF. (1) By Lemma 3.1 (Ch. 1), the fibers are connected. Hence a connected component Y' of Y which contains a point of a fiber contains already the complete fiber, therefore $Y' = \varphi^{-1}(\varphi(Y'))$.

(2) Clearly $Z = \varphi(Y')$.

(3) Trivial

(4) Recall that a submersion $\varphi : Y \to X$ between topological spaces is surjective and has the property that $\varphi^{-1}(T)$ is open if and only if $T \subseteq X$ is open. Now, assume that $s : X \to Y$ is a section, i.e. $\varphi \circ s = Id_X$. Then,

clearly $Z = s^{-1}(\varphi^{-1}(Z))$ for any subset $Z \subseteq X$, therefore Z is open if and only if $\varphi^{-1}(Z)$ is open. In conclusion (4) follows from (5).

(5) We see that if $W \subseteq Y$ is a connected component then $\varphi(W)$ is so is. Effectively, by (2) we know that $W = \varphi^{-1}(\varphi(W))$, so $\varphi(W) \subseteq X$ is closed and clearly is connected because W is connected. Finally, $\varphi(W)$ cannot be strictly contained in a connected component of X, say C, because on that case $W \subsetneq \varphi^{-1}(C)$, due to the surjectivity of φ, implying that $\varphi^{-1}(C)$ is not connected. But, this is impossible because the preimages of closed connected subsets of a submersion are closed connected. In fact, if $D \subseteq X$ is closed connected and $\varphi^{-1}(D) = V_1 \uplus V_2$ for two nonempty closed subsets of Y, then due to the connectedness of the fibers and the fact that $V_i = \varphi^{-1}(\varphi(V_i))$ we see that $W = \varphi(V_1) \uplus \varphi(V_2)$, a contradiction. □

In conclusion, φ sends the connected components of Y onto the connected components of X, inducing the desired bijection between them.

Example 1.2. The conditions in Proposition 1.1 (4), (5) are necessary, as the following example shows. Consider $R = K[X]$, and the nonhomogeneous forcing algebra $A = R[T]/(X^2T - X)$. The minimal primes of $(X^2T - X)$ are (X) and $(XT - 1)$, which are comaximal (since $1 = XT - (XT - 1)$). So by the Chinese Remainder Theorem $A \cong R[T]/(X) \times R[T]/(XT - 1)$ and therefore

$$\operatorname{Spec} A = \operatorname{Spec} k[T] \uplus \operatorname{Spec} K[X,T]/(XT - 1)\,,$$

i.e. a disjoint union of a line over a point and a hyperbola dominating the base. In fact, its image is the pointed affine line, hence dense, because the prime ideals of $K[X]$ not containing are in bijection with the primes of $K[X]_X$, which are exactly the image of $\operatorname{Spec} K[X,T]/(XT-1)_X$, since $K[X,T]/(XT-1) \cong K[X]_X$. But, $\operatorname{Spec} R$ is the affine line which is connected. Note that the element X belongs to the radical of (X^2). Hence $\varphi : \operatorname{Spec} A \to \operatorname{Spec} R$ is surjective, but φ is not submersive due to the fact that $V(X) \subseteq \operatorname{Spec} K[X]$ is not open but by the former facts $\varphi^{-1}(V(X)) = \operatorname{Spec} K[X,T]/(XT - 1)$ is open (for the relation with integral closure see Theorem 6.2 and Remark 6.3).

2. Horizontal and Vertical Components

We describe now the irreducible components of the spectrum of a forcing algebra over an integral base in the ideal case. We will identify prime ideals inside $R[T_1, \ldots, T_n]$ minimal over $(f_1T_1 + \ldots + f_nT_n + f)$ with the minimal prime ideals of the forcing algebra $R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$.

Lemma 2.1. *Let R be a Noetherian domain, $I = (f_1, \ldots, f_n)$ an ideal, $f \in R$, $A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ the forcing algebra for these data and $\varphi : Y \to X$ the forcing morphism. Then the following hold.*

(1) *For $I \neq 0$ there exists a unique irreducible component $H \subseteq \operatorname{Spec} A$ with the property of dominating the base $\operatorname{Spec} R$ (i.e. the image of H is dense). This component is given (inside $R[T_1, \ldots, T_n]$) by*

$$\mathfrak{p} = R[T_1, \ldots, T_n] \cap (f_1T_1 + \ldots + f_nT_n + f)Q(R)[T_1, \ldots, T_n],$$

where $Q(R)$ denotes the quotient field of R.

(2) *All other irreducible components of $\operatorname{Spec} A$ are of the form*

$$V(\mathfrak{q}R[T_1, \ldots, T_n]),$$

for some prime ideal $\mathfrak{q} \subseteq R$ which is minimal over $(f_1, \ldots, f_n, f)$.

(3) *For a minimal prime ideal $\mathfrak{q} \subseteq R$ over $(f_1, \ldots, f_n, f)$ the extended ideal $\mathfrak{q}R[T_1, \ldots, T_n]$ defines a component of $\operatorname{Spec} A$ if and only if $I = 0$ or $I \neq 0$ and $\mathfrak{p} \not\subseteq \mathfrak{q}R[T_1, \ldots, T_n]$.*

Proof. (1). For $I \neq 0$ the polynomial $f_1T_1 + \ldots + f_nT_n + f$ is irreducible (thus prime) in $Q(R)[T_1, \ldots, T_n]$, hence the intersection of this principal ideal with $R[T_1, \ldots, T_n]$ gives a prime ideal in this polynomial ring and therefore in $R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$. The minimality is clear, since it holds in a localization. Because of $\varphi(\mathfrak{p}) = R \cap \mathfrak{p} = 0$, $X = V(0) \subseteq \overline{\varphi(\mathfrak{p})}$ then this component dominates the base. On the other hand, to see the uniquely determine, let $\mathfrak{p}'$ be a minimal prime ideal that is minimal over $(f_1T_1 + \ldots + f_nT_n + f)$ and suppose that $R \cap \mathfrak{p}' = 0$. Let $h \in \mathfrak{p}$. Then there exists $r, s \in R$, $r \neq 0$, and a polynomial $G \in R[T_1, \ldots, T_n]$ such that $rh = sG(f_1T_1 + \ldots + f_nT_n + f)$. This element belongs to $\mathfrak{p}'$ and since $r \notin \mathfrak{p}'$ we deduce $h \in \mathfrak{p}'$. Hence $\mathfrak{p}' = \mathfrak{p}$. Another way to prove this, is noting that there is a bijection between the minimal primes of A not intersecting $R \setminus \{0\}$ and the minimal primes of $K(R) \otimes A$, which is just the zero ideal.

(2). Let $(f_1T_1 + \cdots + f_nT_n + f) \subseteq Q$ be a minimal prime ideal different from $\mathfrak{p}$. Since each $f_i \neq 0$ is not nilpotent, then in the localization $A_{f_i} \cong R_{f_i}[T_1, ..., \check{T}_i, ..., T_n]$ there is only one minimal prime ideal, the zero ideal, defining the horizontal component of $\operatorname{Spec} A_{f_i}$ (as in (1)). Therefore, if $QA_{f_i} \in \operatorname{Spec} A_{f_i}$, then $QA_{f_i} = 0$, because it is minimal on the domain A_{f_i}. But, due to the fact that $A_{f_i} \subseteq K(R) \otimes A$, we get $QK(R) \otimes A = 0$ and thus $Q \cap R = 0$, and then by (1) $Q = \mathfrak{p}$, a contradiction. Hence, $QA_{f_i} \notin \operatorname{Spec} A_{f_i}$ but that implies $f_i \in Q$. Because Q contains the forcing equation we also deduce $f \in Q$. Hence $(f_1, \ldots, f_n, f) \subseteq Q$ and by the minimality condition Q is minimal over the extended ideal $(f_1, \ldots, f_n, f)R[T_1, \ldots, T_n]$. Therefore Q must be the extended ideal of a minimal prime ideal $\mathfrak{q}$ of $(f_1, \ldots, f_n, f)$ in R (the minimal prime ideals above $(f_1, \ldots, f_n, f)R$, above $(f_1, \ldots, f_n, f)R[T_1, \ldots, T_n]$ and above $(f_1, \ldots, f_n, f)A$ are in bijection).

(3). Let $\mathfrak{q}_1 \neq \mathfrak{q}_2$ be minimal prime ideals of $(f_1, \ldots, f_n, f)$ in R, so $\mathfrak{q}_1A$ and $\mathfrak{q}_2A$ are incomparable. Then, $\mathfrak{q}A$ is a minimal prime ideal of $\operatorname{Spec} A$ if and only if $\mathfrak{p} \not\subseteq \mathfrak{q}A$ (since by (ii) we know there are no other possible minimal prime ideals). □

We call $H = V(\mathfrak{p})$ the *horizontal component* of the forcing algebra and the other components $V(\mathfrak{q}_j)$ the *vertical components*. If $I = 0$ there exist only the vertical components which are in bijection with the components of $\operatorname{Spec} R/(f)$.

REMARK 2.2. If $n = 1$, R is a domain and $f_1 \neq 0$, then the horizontal component is just the closure of the graph of the rational function $T = -\frac{f}{f_1}$ view as a subset of $Y = \operatorname{Spec} A$. To see this remember that the graph of $-f/f_1 \in R_{f_1}$ is, by definition, $\operatorname{Spec} R_{f_1}[T]/(T + f/f_1) = \operatorname{Spec} A_{f_1} \cong D(f_1) \subseteq Y$. Then, we need to prove that $H = V(\mathfrak{p}) = \overline{D(f_1)} \subseteq Y$. Let $D(J) \subseteq Y$ a neighborhood of $\mathfrak{p}$, i.e. the ideal $J \nsubseteq \mathfrak{p}$. Suppose that $D(J) \cap D(f_1) = D(J \cdot (f_1)) = \emptyset$, that implies, in particular, that $J \cdot (f_1) \subseteq \mathfrak{p}$, but we know that there exists an element $j \in J \smallsetminus \mathfrak{p}$. Since, $jf_1 \in \mathfrak{p}$ we have $f_1 \in \mathfrak{p}$, which is impossible, because $\mathfrak{p} \cap R = 0$. So, $D(J) \cap D(f_1) \neq \emptyset$, implying that $\mathfrak{p} \in \overline{D(f_1)}$ and thus $V(\mathfrak{p}) \subseteq \overline{D(f_1)}$.

Conversely, let $Q \in D(f_1)$, i.e. $f_1 \notin Q$, we see that $Q \in V(\mathfrak{p})$. In fact, choose a minimal prime ideal $P' \in \operatorname{Spec} A$ contained in Q, then by Lemma 2.1, $P' = \mathfrak{p}$ or $P' = \mathfrak{q}A$ for some prime ideal $\mathfrak{q} \in \operatorname{Spec} R$ minimal over (f_1, f). Now, the second case is impossible because $f_1 \notin \mathfrak{p}$. In conclusion, $D(f_1) \subseteq V(\mathfrak{p})$, and so $\overline{D(f_1)} \subseteq V(\mathfrak{p})$. Hence, $\overline{D(f_1)} = V(\mathfrak{p})$.

REMARK 2.3. If R is a Noetherian domain and $I = (f_1, \ldots, f_n) \neq 0$, then the horizontal component exists and the describing prime ideal $\mathfrak{p}$ has height one in $R[T_1, \ldots, T_n]$. If $\mathfrak{q}$ is a minimal prime ideal over $(f_1, \ldots, f_n, f)$ of height one, then the extended ideal in the polynomial ring has also height one and can therefore not contain the horizontal prime ideal. Therefore such prime ideals yield vertical components.

It is possible that all the $V(\mathfrak{q}_j)$, where $\mathfrak{q}_j \supseteq (f_1, \ldots, f_n, f)$ is a minimal prime ideal, lie on the horizontal component. In this case there exists no vertical component. This happens, for example, if the forcing equation generates a prime ideal, because on that case the horizontal component is the whole space, i.e. $H = V(0) = \operatorname{Spec} A$.

If the forcing equation has a nice factorization inside the polynomial ring $R[T_1, \ldots, T_n]$, then we can describe the horizontal and vertical components more explicitly.

LEMMA 2.4. *Let R be an Noetherian integral domain and A be a forcing algebra over R with forcing equation $h = f_1T_1 + \cdots + f_nT_n + f = dh'$, where $(f_1, \ldots, f_n) \neq 0$, $d \in R$ and where $h' = f_1'T_1 + \cdots + f_n'T_n + f'$ is a prime element in $B = R[T_1, \ldots, T_n]$. Then the horizontal component of* $\operatorname{Spec} A$ *is $V(h')$ and the vertical components of* $\operatorname{Spec} A$ *are $V(\mathfrak{q}A)$, where $\mathfrak{q}$ varies over the minimal prime ideals of (d) in R.*

PROOF. Because h' is a prime element we have

$$(h')R[T_1,\ldots,T_n] = R[T_1,\ldots,T_n] \cap (h)Q(R)[T_1,\ldots,T_n]$$

and this gives by Lemma 2.1 (1) the horizontal component. By Lemma 2.1 (3) we have to show that the minimal prime ideals over (d) correspond to the minimal prime ideals over $J = (f_1,\ldots,f_n,f)$ with the additional property that their extension to $R[T_1,\ldots,T_n]$ does not contain h'.

So let $\mathfrak{q}$ be a minimal prime ideal over (d). Then by Krull's theorem the height of $\mathfrak{q}$ is 1 and hence it is also minimal over $J \subseteq (d)$. Moreover, the height of $\mathfrak{q}R[T_1,\ldots,T_n]$ is also 1. Besides, $h' \notin \mathfrak{q}b$, for otherwise $(0) \subsetneq (h') \subsetneq \mathfrak{q}R[T_1,\ldots,T_n]$ would be a chain of prime ideals of length 2, because $(h') \neq \mathfrak{q}B$, since $(h') \cap R = 0$ and $\mathfrak{q}R[T_1,\ldots,T_n] \cap R = \mathfrak{q}$.

Conversely, let $\mathfrak{q}$ denote a minimal prime ideal of J such that h' does not belong to $\mathfrak{q}R[T_1,\ldots,T_n]$. Assume that $d \notin \mathfrak{q}$. Then because of

$$f_1'd,\ldots,f_nd,f'd \in \mathfrak{q}$$

we get $f_1,\ldots,f_n,f \in \mathfrak{q}$ and hence the contradiction $h' \in \mathfrak{q}R[T_1,\ldots,T_n]$. So we must have $d \in \mathfrak{q}$ and $\mathfrak{q}$ must also be minimal over (d). $\square$

If R is a (not necessarily Noetherian) unique factorization domain (UFD), then there exists always a factorization $h = dh'$ with h' a prime element in B. The minimal prime ideals over (d) are given by the prime factors p of d, and pB has also height 1. Hence the argument of the lemma goes through also in this case. Example 4.7 below shows that a forcing equation does not always have a prime decomposition as in the lemma. Then it is more complicated to determine the vertical components.

Notice that Lemma 2.1(1) also holds in the module case. The following example shows that Lemma 2.1 (2) is not true and the irreducible components in the module case are more complicated.

EXAMPLE 2.5. Consider over $R = K[X,Y]$ the forcing algebra

$$A = R[T_1,T_2]/(XT_1-XY, YT_2-XY) \cong R[T_1,T_2]/\left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} - \begin{pmatrix} XY \\ XY \end{pmatrix}\right).$$

It is easy to see that horizontal component of $\operatorname{Spec} A$ is given by the prime ideal (T_1-Y, T_2-X). The algebra is connected by Lemma 1.1(4), since there is a section sending $T_1 \to Y$ and $T_2 \to X$. The other minimal prime ideals are (X,T_2), (Y,T_1) and (X,Y). But, only the last one is the extension of a prime ideal of the base.

3. Connectedness Results

The following is our main general connectedness result on forcing algebras.

THEOREM 3.1. *Let R be a Noetherian domain, $I = (f_1, \ldots, f_n)$ an ideal $\neq 0$, $f \in R$ and $A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ the forcing algebra for these data. Let $H = V(\mathfrak{p})$ be the horizontal component of* $\operatorname{Spec} A$ *and let $V_j = V(\mathfrak{q}_j)$, $j \in J$, be the vertical components of* $\operatorname{Spec} A$ *according to Lemma 2.1. Let $Z_i = \bigcup_{j \in J_i} V_j$ be the connected components of $\bigcup_{j \in J} V(\mathfrak{q}_j)$. Then* $\operatorname{Spec} A$ *is connected if and only if H intersects every Z_i.*

PROOF. More general, assume that Y is a topological space and $H, V_j \in Y$, where $j \in J$, are connected closed subsets such that Y is the finite union of them. Then, with the same notation for the Z_i, Y is connected if and only if H intersects every Z_i. Efectivelly, one direction is trivial, for the other, assume by contradiction that there exists some Z_j such that $H \cap Z_j = \emptyset$. Then, $Y = Z_j \uplus (\cup_{r \neq j} Z_r \cup H)$, where each of the tow disjoint subsets is a nonempty closed subset due to the fact that we have just finitely many subsets. Note that the fact that H is exactly the horizontal component is not relevant at all. The advantage in making this choice is that we can determine the Z_j just over $\operatorname{Spec} A$, since $V_i \cap V_j \neq \emptyset$ if and only if $\mathfrak{q}_i + \mathfrak{q}_j \subsetneq R$.

□

COROLLARY 3.2. *Let R be a Noetherian domain of dimension* 1*, $I = (f_1, \ldots, f_n)$ an ideal $\neq 0$, $f \in R$ and $A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ the forcing algebra for these data. Let $H = V(\mathfrak{p})$ be the horizontal component of* $\operatorname{Spec} A$ *and let $V_j = V(\mathfrak{q}_j)$, $j \in J$, be the vertical components of* $\operatorname{Spec} A$ *according to Lemma 2.1. Then* $\operatorname{Spec} A$ *is connected if and only if H intersects every V_j.*

PROOF. This follows from Theorem 3.1 since the minimal prime ideals of $I \neq 0$ in a one-dimensional domain are maximal ideals. These maximal ideals form the connected components of $V(I)$. □

Note that this corollary is not true in higher dimension, see Example 4.8 in the next section. We specialize now to the local case.

COROLLARY 3.3. *Let $(R, \mathfrak{m})$ be a local domain, $I = (f_1, \ldots, f_n) \subseteq \mathfrak{m}$ an ideal $\neq 0$, $f \in \mathfrak{m}$ and $A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ the forcing algebra for these data. Let $H = V(\mathfrak{p})$ be the horizontal component of* $\operatorname{Spec} A$ *according to Lemma 2.1. Then* $\operatorname{Spec} A$ *is connected if and only if $\mathfrak{p} + (I, f)B \neq B$ in $B := R[T_1, \ldots, T_n]$.*

PROOF. Let $\mathfrak{q}_j$, $j \in J$, be the minimal prime ideals of (I, f), disregarding whether the $V_j = V(\mathfrak{q}_j B)$ give rise to vertical components of $\operatorname{Spec} A$ or not. Note that $V_j \cap V_i \neq \emptyset$ for all i, j, because $mB \in V_j$, since we are over a local ring. First, suppose that for at least one j we have $V_j = V(\mathfrak{q}_j B) \subseteq H$, that means $\mathfrak{p} \subseteq \mathfrak{q}_j B$, so $\mathfrak{p} + (I, f)B \subseteq mB$, and $\mathfrak{p} + \mathfrak{q}_i B \subseteq mB$ for all i. But, that means $H \cap V_i \neq \emptyset$, for all i, therefore by Theorem 3.1 $\operatorname{Spec} A$ is connected. So under this assumption the two properties holds.

So suppose next that $V_j \not\subseteq H$ for all j, meaning that all V_j are vertical components of $\operatorname{Spec} A$. The union of the subsets $V(\mathfrak{q}_j B)$ inside the local ring $\operatorname{Spec} R$ form just one connected component by a previous comment because any pair of them has nonempty intersection. Hence, there is exactly one Z in the notation of Theorem 3.1. By this theorem, $\operatorname{Spec} A$ is connected if and only if $H \cap Z \neq \emptyset$. Because of $Z = V(\cap_j \mathfrak{q}_j B)$ this is equivalent to $\mathfrak{p} + (I, f)B \neq B$, since $\cap_j \mathfrak{q}_j = \operatorname{rad}\,((I, f)B)$. □

EXAMPLE 3.4. Let $R = K[X, Y]_{(X,Y)}$, let $A = R[T]/(XYT - X)$. The horizontal component is $V(YT - 1)$ and the only vertical component is $V(X)$. Because they intersect (or because $(YT - 1, X) \neq (1)$) the forcing algebra is connected. However, we have $(YT - 1, \mathfrak{m}) = (1)$, that means, $H \cap V(mA) = \emptyset$, so the connectedness over a local ring does not imply that the horizontal component meets the fiber over the maximal ideal.

4. Connectedness over Unique Factorization Domains

We deal now with the case where R is a UFD. Note that if R is a UFD, then $B = R[T_1, \ldots, T_n]$ is factorial as well. So if $h = f_1T_1 + \ldots + f_nT_n + f \in B$ is a forcing equation, then one can factor out a greatest common divisor of all the coefficients $f_1, \ldots, f_n$ and f, say d, and obtain a representation of h as a product of an element d in R and an irreducible polynomial $h' = (f_1/d)T_1 + \ldots + (f_n/d)T_n + (f/d)$ in B (for $n \geq 1$), which generates a prime ideal because B is a UFD. This hypothesis appeared already in Lemma 2.4 and is also crucial in the following sufficient condition for connectedness.

COROLLARY 4.1. *Let R be a Noetherian domain, $B := R[T_1, \ldots, T_n]$, and let*

$$h := f_1T_1 + \ldots + f_nT_n + f = d(f_1'T_1 + \ldots + f_n'T_n + f')$$

be a forcing equation such that $h' := f_1'T_1 + \ldots + f_n'T_n + f'$ is a prime polynomial. Suppose that $(f_1, \ldots, f_n) \neq 0$ and $(f_1', \ldots, f_n')$ is not contained in any minimal prime ideal of (d). Then $\operatorname{Spec} A$ is connected, where $A = B/(h)$.

PROOF. By Lemma 2.4, the horizontal component of $\operatorname{Spec} A$ is $V(h')$ and the vertical components correspond to the minimal prime ideals $\mathfrak{q}$ over (d). We will show that $V(\mathfrak{q})$ intersects the horizontal component.

This can be established after the base change $R \to \kappa(\mathfrak{q})$. Now at least one of the f_i' becomes a unit in $\kappa(\mathfrak{q})$ and therefore h' is not a unit over $\kappa(\mathfrak{q})$. So $\kappa(\mathfrak{q})[T_1, \ldots, T_n]/(h') \neq 0$. □

Note also that if d is a unit, then $h = h'$ is a prime polynomial by assumption and then the forcing algebra is integral, hence connected anyway.

Now, we shall deduce a Corollary in the case that R is a UFD. In this kind of rings we can define a greatest common divisor of a finite set of elements $a_1, \ldots, a_m$, denoted by $\gcd(a_1, \ldots, a_m)$, using the prime factorization, and it is well defined up to a unit in R and defined as the unity in R in the case that the elements have no irreducible common factor.

LEMMA 4.2. *Let R be a UFD, $f_1, \ldots, f_n, f \in R$ with some $f_i \neq 0$ and let d be a greatest common divisor of $f_1, \ldots, f_n$ and f and write*

$$h = f_1T_1 + \ldots + f_nT_n + f = d(f_1'T_1 + \ldots + f_n'T_n + f')$$

where $f_i' = f_i/d$. Then $h' = f_1'T_1 + \ldots + f_n'T_n + f'$ is an irreducible polynomial and describes the horizontal component of $\operatorname{Spec} R[T_1, \ldots, T_n]/(h)$.

PROOF. Suppose we have a factorization $h' = h_1h_2$ in $B = R[T_1, \ldots, T_n]$. Then one of the h_i can not contain any variable T_j, say h_1, thus $h_1 \in R$. Therefore, h_1 divides each $f_1/d, \ldots, f_n/d, f/d$ and therefore it is a unit in R, because these elements have no common irreducible factors. Thus h' is an irreducible polynomial and hence a prime element (B is also a UFD) and therefore by the correspondance between the prime ideals of the localization $K(R) \otimes B$ and the prime ideals of B not intersecting $R \setminus \{0\}$ and by Lemma 2.4(1), we have that $\mathfrak{p} = (dh')K(R) \otimes B \cap B = (h')K(B) \otimes B \cap B = (h')B$. $\square$

COROLLARY 4.3. *Let R be a UFD, $f_1, \ldots, f_n, f \in R$ with some $f_i \neq 0$ and let d be a greatest common divisor of $f_1, \ldots, f_n$ and f. Let*

$$h = f_1T_1 + \ldots + f_nT_n + f = d(f_1'T_1 + \ldots + f_n'T_n + f')$$

be the forcing equation and let $d = p_1 \cdots p_k$ be a prime factorization of d. Suppose that $\{1, \ldots, k\} = \biguplus_{i \in I} J_i$ such that the $\bigcup_{j \in J_i} V(p_j)$ are the connected components of $V(d)$. Then $\operatorname{Spec} B/(h)$ is connected if and only if for every i there exists some $j \in J_i$ such that $(f_1'T_1 + \ldots + f_n'T_n + f', p_j) \neq (1)$.

PROOF. By Lemma 2.4 and Lemma 4.2, $\mathfrak{p} = (f_1'T_1 + \ldots + f_n'T_n + f')$ describes the horizontal component and every vertical component corresponds to a prime factor of d. Hence the statement follows directly from Theorem 3.1. $\square$

The condition that $(f_1'T_1 + \ldots + f_n'T_n + f', p) \neq (1)$ is true if p does not divide all $f_1', \ldots, f_n'$ or if f' is not a unit modulo p. But, none of this condition is equivalent to the one given in the corollary as the following examples shows.

EXAMPLE 4.4. Let $R = K[X, Y]$. For $h = X(XT+Y)$ we have $d = X = p$. Because Y is not a unit modulo X, the forcing algebra is connected. For $h = X(XT + X + 1)$ we also have $d = X = p$. Now, $X + 1$ is a unit modulo X and the forcing algebra is not connected. However for $h = X(YT + X + 1)$, we have that $X + 1$ is a unit modulo X but the forcing algebra is connected. For $h = XY(XT_1 + YT_2 + f')$ we have $d = XY$, but $f_1' = X$, $f_2' = Y$ do not have a common prime factor and hence the forcing algebra is connected. On the other hand, for $h = X(XT + Y)$, the forcing algebra is connected but X divides $f_1 = X$.

COROLLARY 4.5. *Let R be a UFD, $B = R[T_1, \ldots, T_n]$, $h = f_1T_1 + \ldots + f_nT_n + f$ a forcing equation and $A = B/(h)$. Let d be a greatest common divisor of $f_1, \ldots, f_n$ and f, and assume that at least one of the $f_i \neq 0$. Suppose that $\gcd(d, f_1/d, \ldots, f_n/d) = 1$, then $\operatorname{Spec} A$ is connected. Moreover, if R is a principal ideal domain then the condition $\gcd(d, f_1/d, \ldots, f_n/d) = 1$ is equivalent to $\operatorname{Spec} A$ being connected.*

PROOF. We write $h = dh'$, where $h' = f_1'T_1 + \ldots + f_n'T_n + f'$ where $f_i' = f_i/d$ and $f' = f/d$. Thus, the first part follows directly from Corollary 4.1, because the hypothesis clearly implies that $(f_1', ..., f_n')$ is not contained in any minimal prime ideal of (d). For other proof, let p be a prime factor of d. Then p does not divide some f_i'. But then $f_1'T_1 + \ldots + f_n'T_n + f'$ is not a unit modulo p, and the condition of Corollary 4.3 holds even for every p.

Finally, we assume that R is a principal ideal domain and that

$$(d, f_1', \ldots, f_n') = (e),$$

where $e \in R$ is not a unit. Let $p \in R$ be a prime element dividing e. We still work with the factorization $h = dh'$, where h' is irreducible and describes the horizontal component. The elements $f_1/d, \ldots, f_n/d, f/d$ do not have a common prime factor, hence p does not divide f/d. Therefore in the field $R/(p)$ the element f/d becomes a unit u and the polynomial h' becomes $0T_1 + \ldots + 0T_n + u$. Therefore the horizontal component $V(h')$ and the vertical component $V(p)$ are disjoint and the forcing algebra is not connected by Corollary 3.2. □

REMARK 4.6. The previous corollary suggests a deep relation between the topology of the quotients of UFD by principal ideals and the arithmetic of the generators of these ideals. Therefore, it makes sense to define the notion of *connected numbers* in UFDs B as numbers t such that $\operatorname{Spec} B/(t)$ is a connected topological space, and to explore the arithmetic-topological connections coming from this new definition.

EXAMPLE 4.7. We consider the domain $R = K[X, Y, Z]/(Z^2 - XY)$. This is not a UFD, since $Z^2 = XY$ can be written in two ways as a product of irreducible factors. Accordingly, the rational function $q = \frac{Z}{X} = \frac{Y}{Z}$ is defined on $D(X, Z)$, and (X, Z) is a prime ideal of height one not given by one element. We look at the forcing algebra

$$B = R[T]/(XT - Z).$$

It is elementary to see that the element $XT - Z$ is irreducible in $R[T]$, but not prime, because $Z(XT - Z) = X(ZT - Y)$, but neither X nor $ZT - Y$ belongs to $(XT - Z)$. The minimal prime ideals over $(XT - Z)$ are $\mathfrak{p} = (XT - Z, ZT - Y)$. Effectively, due to $ZT - Y = (Z/X)(XT - Z)$, we deduce that $ZT - Y \in \mathfrak{p}$ and we can see that $R[T]/(XT - Z, ZT - Y) \cong R[X, Z, T]/(XT - Z)$, which is a domain, and so $(XT - Z, ZT - Y) \subseteq \mathfrak{p}$ is a prime ideal who also dominates the base, then they are equal. In conclusion, it describes the horizontal component in the spectrum of the forcing algebra B, corresponding to the closure of the graph of the rational function q (see Remark 2.2). Finally, the vertical component is $(X, Z)R[T]$. Because of $(X, Z) + (XT - Z, ZT - Y) = (X, Y, Z)$, these two components intersect and therefore the forcing algebra is connected.

Example 4.8. The condition of being a principal ideal domain for R in the last part of Corollary 4.5 is necessary, as the following example shows (see also Figure 1, for the case $K = \mathbb{R}$). With the notation from above we consider the following setting: $R := K[X, Y]$, $B = R[T]$, $h = X^2YT - XY = XY(XT - 1)$ and $A := B/(h)$. Clearly, $d = \gcd(X^2Y, XY) = XY$, $f_1 = X^2Y$ and $\gcd(d, f_1/d) = \gcd(XY, X) = X \neq 1$. Besides, as Lemma 2.1 or Lemma 2.4 shows, the irreducible components of $\operatorname{Spec} A$ are the horizontal component $V((XT - 1)A)$ (red hyperbolic surface) and the vertical components $V(XA)$ (blue plane) and $V(YA)$ (green plane). Furthermore, $V(XA) \cap V(YA) = V((X, Y)A) \neq \emptyset$, so the two vertical components are connected. Because of $V(YA) \cap V((XT - 1)A) = V((Y, XT - 1)A) \neq \emptyset$ (note also that $V(XA) \cap V((XT - 1)A) = V((X, XT - 1)A) = \emptyset$) the condition of Theorem 3.1 (or Corollary 4.3) is fulfilled and hence $\operatorname{Spec} A$ is connected. However, the condition of a greatest common divisor in Corollary 4.5 does not hold.

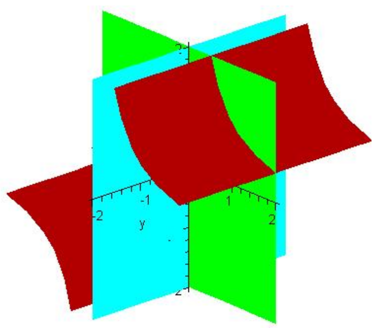

Figure 1. Corresponding to $(\operatorname{Spec} A)_{\mathbb{R}}$.

5. Local Properties

An interesting question is whether the connectedness of $Y = \operatorname{Spec} A$ is a local property over the base $X = \operatorname{Spec} R$. Specifically, is it true that Y is a connected space if and only if X is connected and for every $\mathfrak{p} \in X$, $\operatorname{Spec} A_{\mathfrak{p}}$ is connected, where $A_{\mathfrak{p}}$ denotes the localization of A at the multiplicative system $R \setminus \mathfrak{p}$ (considered in A). The next theorem gives a positive answer to the "if" part of this question in general. For a forcing algebra, the converse holds for a one-dimensional domain, but neither over a reducible curve nor over the affine plane.

THEOREM 5.1. *Let* $\psi : R \to A$ *be a ring homomorphism. Set* $X := \operatorname{Spec} R$ *and* $Y := \operatorname{Spec} A$*. Suppose that* X *is a connected space and that for all* $\mathfrak{p} \in X$*,* $\operatorname{Spec} A_{\mathfrak{p}}$ *is connected, where* $A_{\mathfrak{p}} := A_{R \setminus \mathfrak{p}}$*, that is,* Y *is locally (over the base) connected. Then* Y *is connected.*

PROOF. We first show that we can assume that ψ is injective: first, note that for any minimal prime $\mathfrak{p} \in X$, the space $\operatorname{Spec} A_{\mathfrak{p}}$ is not empty, because it is connected (our convention is that the empty set is not connected). Let $Q \in \operatorname{Spec} A_{\mathfrak{p}}$ be a prime ideal, then $\psi^{-1}(Q)$ is a prime ideal of R contained in $\mathfrak{p}$, because $\psi^{-1}(Q) \cap (R \setminus \mathfrak{p}) = \emptyset$, moreover it is equal to $\mathfrak{p}$ in view of the minimality of $\mathfrak{p}$. Therefore, for any minimal prime in R, there exists a prime ideal Q in A lying over it. Therefore for any $a \in \ker \psi$, we know that $\psi(a) = 0 \in \cap_{Q \in Y} Q$ and then $a \in \cap_{Q \in Y} \psi^{-1}(Q) \subseteq \cap_{\mathfrak{p} \in \min R} \mathfrak{p} = \operatorname{nil} R$, that is $\ker \psi \subseteq \operatorname{nil} R$.

In consideration of this it is enough to reduce to the case of R being reduced. For this reduction consider the natural homomorphism $\psi_{\mathrm{red}} := R_{\mathrm{red}} \to A_{\mathrm{red}}$ induced by ψ, killing the nilpotent elements. Now, our hypothesis of locally (over the base X) connected and the conclusion holds for Y if and only if if holds (over the base $X_{\mathrm{red}} = \operatorname{Spec}(A_{\mathrm{red}})$) for $Y_{\mathrm{red}} := \operatorname{Spec}(A_{\mathrm{red}})$. In fact, clearly $X \approxeq X_{\mathrm{red}}$ and $Y \approxeq Y_{\mathrm{red}}$ as topological spaces, besides, for any $\mathfrak{p} \in X_{\mathrm{red}}$, $(A_{\mathrm{red}})_{\mathfrak{p}} \approxeq A_{\mathfrak{p}}/(\operatorname{nil} A)A_{\mathfrak{p}}$ and $(\operatorname{nil} A)A_{\mathfrak{p}} \subseteq \operatorname{nil}(A_{\mathfrak{p}})$, hence $\operatorname{Spec}((A_{\mathrm{red}}))_{\mathfrak{p}} \approxeq \operatorname{Spec}(A_{\mathfrak{p}}/\operatorname{nil} A_{\mathfrak{p}}) \approxeq \operatorname{Spec} A_{\mathfrak{p}}$. In conclusion, it is enough to prove the theorem in the reduced case for injective ψ.

Now, we assume that Y is not connected, which is equivalent to say that there exists nontrivial idempotents $e_1, e_2 \in A$ with $e_1 + e_2 = 1$, $e_1 e_2 = 0$ and $e_1, e_2 \neq 0, 1$. Set $J_i = \operatorname{Ann}_R(e_i)$ for $i = 1, 2$. We claim that $J_1 + J_2 \subsetneq R$. Otherwise there exists $y_i \in J_i$ such that $y_1 + y_2 = 1$, and then $y_1 y_2 = y_1 y_2 (e_1 + e_2) = y_2(y_1 e_1) + y_1(y_2 e_2) = 0 + 0 = 0$. Therefore $X = V(y_1) \uplus V(y_2)$, that is, we can write X as a disjoint union of two closed subsets, which implies in view of the connectedness of X that one of these closed subsets is empty, or what is the same, one of the y_i is a unit. Hence, $e_i = y_i^{-1}(y_i e_i) = y_i^{-1} 0 = 0$, a contradiction.

So let $J_1 + J_2 \subseteq P$ be a prime ideal. By assumption, $A_{R \setminus \mathfrak{p}}$ is connected, hence either e_1 or e_2 become 0 in this ring. This means (in the first case) that there exists $s \in R \setminus \mathfrak{p}$ such that $s e_1 = 0$ in A. But then we get the contradiction $s \in J_1$. In conclusion, Y is a connected space. □

We deal next with the one-dimensional case.

COROLLARY 5.2. *Suppose that* R *is a Noetherian domain of dimension 1. Let* $I = (f_1, \ldots, f_n) \neq 0$ *be an ideal,* $f \in R$ *an element and* $A = R[T_1, \ldots, T_n]/(f_1 T_1 + \ldots + f_n T_n + f)$ *the forcing algebra for these data. Then*

Spec A *is connected if and only if* Spec A *is locally connected, i.e. for every prime ideal* $\mathfrak{p} \in \operatorname{Spec} R$ *is* $A_{R\setminus\mathfrak{p}}$ *connected.*

PROOF. The global property follows from the local property by Theorem 5.1. So suppose that Spec A is connected. By the assumption $I \neq 0$ we know that a horizontal component exists. Hence the fiber over the generic point (0) is nonempty, thus connected by Lemma 3.1 (Ch. 1). The connectedness of Spec A means by Corollary 3.2 that the horizontal component meets every vertical component. The vertical components of Spec $A_{\mathfrak{q}}$ over Spec $R_{\mathfrak{q}}$ for a maximal ideal $\mathfrak{q}$ in Spec R are empty or $V(\mathfrak{q}A_{\mathfrak{q}})$, and in the second case the ideal $\mathfrak{p}'$ defining the horizontal component of Spec $A_{\mathfrak{q}}$ is not contained in $\mathfrak{q}A_{\mathfrak{q}}$, therefore the same holds on Spec A, because the horizontal component here is just the intersection of $\mathfrak{p}'$ with A, so $V(\mathfrak{q}A)$ is a vertical component of Spec A. By the intersection condition the horizontal component and this vertical component $V(\mathfrak{q}A)$, (if it exists) intersect on Spec A but that implies, on this particular case, that they intersect also in the localization at $R \setminus \mathfrak{q}$, so Spec $A_{\mathfrak{p}}$ is connected. □

The following example shows that for a non-integral one-dimensional base ring, connectedness is not a local property.

EXAMPLE 5.3. Let $R = K[X,Y]/(XY(X+Y-1))$. Its spectrum has three line components forming a triangle meeting in $(0,0)$, $(1,0)$ and $(0,1)$. Consider the forcing algebra

$$A = R[T]/((Y+X^2)T - X(X+Y-1))\,.$$

Here we will identify the closed subsets of Spec R with the corespondent affine subvarieties of $V(XY(X+Y-1)$. Its spectrum consists in a horizontal line H_1 over $X = 0$, a horizontal line H_2 and one (or two) vertical components over $X+Y=1$, depending on the number of different roots over K of the polynomial $X^2 - X + 1$, a vertical line V over $X = Y = 0$ and the graph G of the rational function $(X-1)/X$ over $Y = 0$. Because of $G \cap H_2 = \{(1,0,0)\}$, $H_1 \cap H_2 = (0,1,0)$ and $H_1 \cap V = (0,0,0)$, the forcing algebra A is connected. However, the localization of the forcing algebra at (X,Y) is not connected, because the connecting component H_2 is missing (the two connected components are $V \cup H_1$ and G).

COROLLARY 5.4. *Suppose that R is a Dedekind domain.*

Let $I = (f_1, \ldots, f_n)$ be an ideal, $f \in R$ an element inside the radical of I and

$$A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$$

be the forcing algebra for these data. Then the following are equivalent.

(1) Spec A *is connected.*
(2) Spec A *is locally connected, i.e. for every prime ideal* $P \in \operatorname{Spec} R$ *is* $A_{R\setminus P}$ *connected.*

(3) $f \in I$.

Proof. The equivalence between (1) and (2) follows from Corollary 5.2. To see that (2) implies (3) we may assume that R is local, (i.e. a discrete valuation domain) because $f \in I$ if and only if for any prime $\mathfrak{q} \subseteq R$ $f/1 \in IR_{idealq}$. Let p be a generator of its maximal ideal. We may assume at once that $I \neq 0$, because else $f = 0$ due to the radical assumption, and also that all f_i and f are not 0. In fact, if $f_i = 0$ we can change the forcing algebra A by a simpler one A' omitting the variable T_i, this without changing any property, because by Lemma 1.1(4) $A = A'[T_i]$ is connected if and only if A' is connected. We write $f_i = u_i p^{k_i}$ and $f = up^k$ with units u_i, u. Assume that $f \notin I$. Then $k < \min(k_1, \ldots, k_n)$. We write the forcing equation as

$$p^k(u_1 p^{k_1-k} T_1 + \ldots + u_n p^{k_n-k} T_n + u)\,,$$

where the exponents $k_i - k$ are all positive. Because of the radical assumption we have $k \geq 1$. But then the forcing algebra has the two components $V(p)$ and $V(u_1 p^{k_1-k} T_1 + \ldots + u_n p^{k_n-k} T_n + u)$ which are disjoint. The other direction follows from Proposition 1.1 (4), since (3) is equivalent to the existence of a section. □

For a non-normal one-dimensional domain this equivalence can never be true because of Corollary 6.4 below. The next trivial example shows that this statement is also not true without the radical assumption and at the same time shows that the condition $I \neq 0$ in Corollary 5.2 is necessarily.

Example 5.5. For $R = K[X]$, the forcing algebra $A = K[X,T]/(0T - X) \cong K[T]$ is connected. But, since $(K[X] \smallsetminus 0)^{-1}A = K(X)[T]/(-X) = 0$, it follows that $\operatorname{Spec}(A_{K[X] \smallsetminus \{0\}}) = \emptyset$, being not connected. Hence, $\operatorname{Spec} A$ is not locally connected.

In higher dimension, even for a factorial domain, the converse of Theorem 5.1 is also not true.

Example 5.6. We continue with Example 4.8 (see Figure 2 for $K = \mathbb{R}$) i.e. $R := K[X,Y]$ is the ring of polynomials in two variables, $B := R[T]$, $h := X^2YT - XY$ and $A := B/(h)$. The morphism $\operatorname{Spec} A \to \operatorname{Spec} R$ is surjective, since $XY \in \operatorname{rad}(X^2Y)$. We know already that $\operatorname{Spec} A$ has the three irreducible components $V(X)$, $V(Y)$ and $V(XT - 1)$ that it is connected.

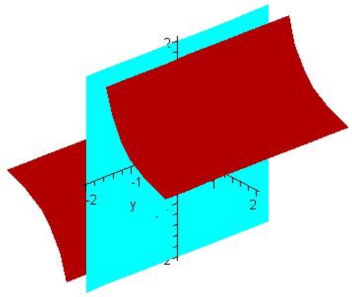

Figure 2. Corresponding to $(\operatorname{Spec} A_{\mathfrak{p}})_{\mathbb{R}}$.

However, if we localize A in $\mathfrak{p} := (X) \in \operatorname{Spec} R$, that is, if we consider the ring $A_{\mathfrak{p}} = A_{R \setminus \mathfrak{p}}$, then $\operatorname{Spec} A_{\mathfrak{p}}$ has just two irreducible components, namely, $V((X)A_{\mathfrak{p}})$ (blue plane) and $V((XT+1)A_{\mathfrak{p}})$ (red hyperbolic surface), because the minimal primes of $A_{\mathfrak{p}}$ are just $(X)A_{\mathfrak{p}}$ and $(XT+1)A_{\mathfrak{p}}$, since the remaining minimal prime ideal (Y) meets $R \setminus \mathfrak{p}$, since $Y \in R \setminus \mathfrak{p}$. Moreover, these two irreducible components are disjoint, because $V((X)A_{\mathfrak{p}})) \cap V((XT-1)A_{\mathfrak{p}}) = V((X, XT-1)A_{\mathfrak{p}}) = V((1)) = \emptyset$. In conclusion, $\operatorname{Spec} A_{\mathfrak{p}}$ is not connected. For getting a better intuition compare Figure 2 with Figure 1.

6. Integral Closure and Connectedness

In this final section we relate the integral closure of an ideal to the universal connectedness of the forcing morphism. For a Noetherian domain, there exists a (discrete) valuative criterion for the integral closure: The containment $f \in \overline{I}$ holds if and only if for all ring homomorphisms $\theta : R \to D$ to a discrete valuation domain D we have $\theta(f) \in ID$, see [**27**, Theorem 6.8.3].

Definition 6.1. Let $\varphi : Y \to X$ be a morphism between affine schemes. We say that φ is a *universally connected* if $W \times_X Y$ is connected for any affine Noetherian change of base $W \to X$, with W connected.

Now, we prove a criterion for belonging to the integral closure in terms of the universal connectedness of the corresponding forcing morphism.

Theorem 6.2. *Let A be a forcing algebra over a Noetherian ring R and $\varphi : Y := \operatorname{Spec} A \to X := \operatorname{Spec} R$ the corresponding forcing morphism. Then the following conditions are equivalent:*

1. *f belongs to the integral closure of I, i.e. $f \in \overline{I}$.*
2. *φ is a universal submersion.*
3. *φ is universally connected.*
4. *$W \times_X Y$ is connected for all change of base of the form $W = \operatorname{Spec} D$, where D is a discrete valuation domain.*

Proof. (1) $\Rightarrow$ (2). Recall that a submersion is universal if it remains a submersion under Noetherian change of base. Due to the valuative criterion for the integral closure (see [**27**, Theorem 6.8.3]) and the fact that (2) can be checked after change of base to a discrete valuation domain D (see [**16**, Remarque 2.6]), we can assume that $R = D$ and that $f \in I$ and we have to prove that $\varphi : \operatorname{Spec} A \to \operatorname{Spec} D$ is a submersion. But $f \in I$ if and only if there exists a section $s : \operatorname{Spec} D \to \operatorname{Spec} A$, i.e. $\varphi \circ s = \mathrm{Id}_{\operatorname{Spec} D}$. But, it implies that φ is a submersion (see proof of Proposition 1.1(4)).

(2) $\Rightarrow$ (3). Let $W \to X$ be an affine Noetherian connected change of base, then since $\varphi_W : W \times_X Y \to W$ is a submersion, by Proposition 1.1 (5) $W \times_X Y$ is connected.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). Let $W = \operatorname{Spec} D \to X$ be a change of base, where D is a discrete valuation domain. By the valuative criterion for integral closure it is enough to show that $f \in ID$. First, note that $f \in \operatorname{rad}(ID)$, which is equivalent to say that $\varphi_W : W \times_X Y \to W$ is surjective. If the fiber over $\mathfrak{p} \in W$, $(\operatorname{Spec} \kappa(\mathfrak{p})) \times_X Y$ were empty, then the morphism

$$\operatorname{Spec} \kappa(\mathfrak{p})[[x]] \longrightarrow \operatorname{Spec} \kappa(\mathfrak{p}) \longrightarrow W \longrightarrow X$$

would yield a contradiction, since then the pull-back $(\operatorname{Spec} \kappa(\mathfrak{p})[[x]]) \times_X Y$ would be empty (hence not connected) and $\kappa(\mathfrak{p})[[x]]$ is a discrete valuation domain. Thus $f \in \operatorname{rad}(ID)$ and by Corollary 5.4 the connectedness of $W \times_X Y$ is equivalent to $f \in ID$. □

Remark 6.3. This Theorem results to be very useful in proving that an specific element does not belong to the integral closure of an ideal, avoiding nontrivial computations. In fact, going back to Example 1.2 we know that φ is not a submersion, therefore X does not belong to the integral closure of (X^2), which is not trivial to see just by doing computations, while proving that φ is not a submersion follows directly from the clear fact that Spec A is not connected.

Now, we prove a corollary of this theorem characterizing the property that a fraction belongs to the integral closure (or normalization) of an integral domain.

Corollary 6.4. *Let R be a Noetherian integral domain, $K = Q(R)$ its field of fractions. Let $r/s \in K$ with $s \neq 0$, let $A = R[T]/(sT+r)$ be the forcing algebra and $\varphi : Y := \operatorname{Spec} A \to X := \operatorname{Spec} R$ the corresponding morphism. Then r/s is integral over R if and only if φ is universally connected.*

Proof. Suppose that $r/s \in K(R)$ is integral over R. Then, there exists an equation of the form

$$(r/s)^n + a_1 (r/s)^{n-1} + \cdots + a_n = 0,$$

for $a_i \in R$. Thus, after multiplying by s^n, we getting

$$r^n + a_1 s r^{n-1} + \cdots + s^n a_n = 0,$$

where $a_i s^{n-i} \in (s)^i$. So, $r \in \overline{(s)}$. The converse in analog. In conclusion, r/s is integer over R if and only if $r \in \overline{(s)}$. Finally, by the previous theorem $r \in \overline{(s)}$ if and only if φ is universally connected. $\square$

In our final example we show that a forcing algebra over a non-normal curve might be connected but not universally connected. In fact the pull-back to the normalization is already not connected.

Example 6.5. Let K be a field and consider the ring-homomorphism $K[u,v] \to K[x]$, $u \mapsto x(x-1)$, $v \mapsto x^2(x-1)$. The kernel of this is $(u^3 - uv + v^2)$. Let $R = K[u,v]/(u^3 - uv + v^2)$. Since $x^2 - x - u = 0$, the extension $R \hookrightarrow K[X]$ is integer, but $K[X]$ is integrally closed, therefore the integral closure (or normalization) of R is $K[X]$. We consider the forcing algebra $A = R[T]/(vT - u)$. It consists of a horizontal component given by $V(vT + u, vT^3 - T + 1)$ (check that this is a prime ideal) (red twisted hyperbola) and the vertical component $V(u,v)$ (black middle vertical line). They intersect in $V(u,v,T-1)$, hence the forcing algebra is connected (see Figure 3).

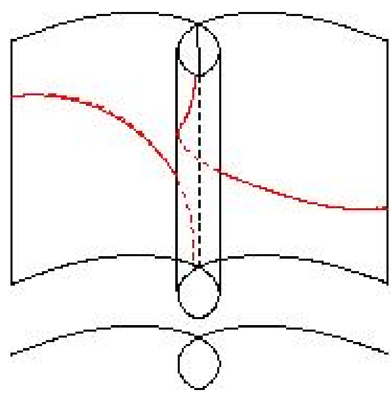

Figure 3

When we pull-back this situation to the normalization we get

$$A' = K[x][T]/(x^2(x-1)T + x(x-1)) \cong K[x][T]/(x(x-1)(xT+1)) \,.$$

Now we have one horizontal component and two vertical components (gray and black lines), and the horizontal (red) hyperbola meets exactly one of them, hence this forcing algebra is not connected by Corollary 3.2 (see Figure

4). Heuristically we get Figure 3 from Figure 4 by turning the plane over itself as the base line indicates and identifying the two vertical lines.

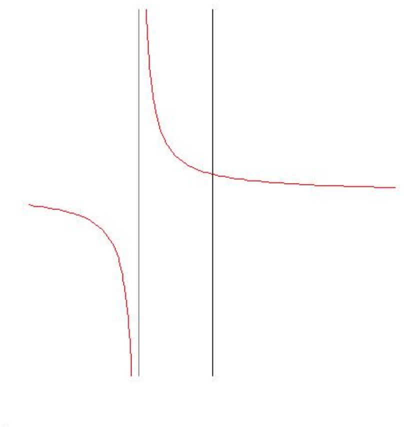

Figure 4

CHAPTER 3

Normality and Related Properties

1. Irreducibility

Here we shall see that if A is a forcing algebra over a Noetherian integral domain such that $\mathrm{ht}(f, f_1, ..., f_n) \geq 2$, where $\{f_1, ..., f_n, f\}$ is the forcing data, then A is an irreducible ring (i.e. A has just one minimal prime).

THEOREM 1.1. *Let R be a Noetherian integral domain;*

$$A = R[T_1, ..., T_n]/(f_1T_1 + \cdots + f_nT_n + f);$$

$h = f_1T_1 + \cdots + f_nT_n + f$, where $f_1, ..., f_n, f \in R$ and $J = (f, f_1, ..., f_n)$. Assume that $\mathrm{ht}J \geq 2$, then A is an irreducible ring.

PROOF. By Lemma 2.1(2) (Ch. 2), it is enough to see that for any minimal prime $\mathfrak{q} \in R$ of J, $\mathfrak{q}B$ is not minimal over (h), because on that case, A has just the horizontal component, and therefore is irreducible.

Let $\mathfrak{q} \in R$ be minimal over J. Then, by Remark 1.1 (Ch. 1),

$$\mathrm{ht}\mathfrak{q}B \geq \mathrm{ht}\mathfrak{q} \geq \mathrm{ht}J \geq 2.$$

Therefore $\mathfrak{q}B$ is not minimal over (h), since by Krull's Principal Ideal Theorem the minimal primes over a principal ideal have height smaller or equal than one. □

2. (Non)Reduceness

In this section we study the (non)reducedness of forcing algebras over a reduced base ring R. First, for a base field k Lemma 3.1 shows that any forcing algebra is isomorphic to a ring of polynomials over k or the zero algebra, therefore it is reduced.

Now, if R is a local ring, let us first stay an elementary remark concerning a generalization of the Monomial Conjecture (MC) (see Ch. 1, §4) in dimension one.

In dimension one (CM) just says that if $x \in m$ does not belong to any minimal prime ideal of R then $x^n \notin (x^{n+1})$ for all nonnegative integer n. In the next remark we will prove a generalization of this fact for a quasi-local ring, that is, not necessarily Noetherian.

REMARK 2.1. Let (R, m) be a quasi-local ring and $x \in m$. Then, there exists a positive integer n such that $x^n \notin (x^{n+1})$, if and only if x is a nilpotent or a unit.

In fact, one direction is trivial, for the other one, assume that x is neither nilpotent nor a unit and that there exists $n \in \mathbb{N}$ and $y \in R$ such that $x^n = yx^{n+1}$, thus $x^n(1-yx) = 0$, but $1-yx \notin m$, therefore it is a unit, then $x^n = 0$, which is a contradiction.

As we mention on section 4 of Chapter 1 (see [**20**, Theorem 1]) the DCS in dimension one is equivalent to the fact that for any parameter $x \in S$, for a one dimensional local ring S, and any power $n \in \mathbb{N}$, $x^n \notin (x^{n+1})$. Clearly, this is a particular case of the previous Remark.

EXAMPLE 2.2. Let (R, m) be a quasi-local reduced ring, which is not a field, and $f \in m \setminus \{0\}$. Then, the trivial forcing algebra $A := R/(f^2)$ is non-reduced because clearly $\overline{f} \in \text{nil}A$ and by the previous Remark $\overline{f} \neq A$. So, there are always non-reduced forcing algebras over quasi-local reduced base rings, which are not a field.

Now, we want to study in which generality we can guarantee the existence of an element $f \in R$ such that $f \notin (f^2)$. Let us assume that R is Noetherian. Then, the following Proposition gives a compact characterization of the fact that any element $f \in R$ belongs to (f^2).

PROPOSITION 2.3. *A noetherian ring R is the finite direct product of fields if and only if any element $f \in R$, holds that $f \in (f^2)$.*

PROOF. On direction is clear.

For the other, let us assume, by contradiction, that R is a Noetherian ring which is not a finite product of fields. We want to prove that there is an element $f \in R$ such that $f \notin (f^2)$. In fact, we can reduced to the case of $\text{Spec}R$ connected, because if $\text{Spec}R$ is not connected then, due to the Noetherian hypothesis, we can write $\text{Spec}R = V(Q_1) \uplus \cdot \uplus V(Q_s)$, where $V(Q_j) \cong \text{Spec}(R/Q_s)$ are the connected components of $\text{Spec}R$. Hence, by the Chinese Remainder Theorem, $R \cong \prod_{i=1}^{s} R_i/Q_i$ and by the previous assumption at least one of the R_i/Q_i is not a field. So, it is enough to find an $f_i \in R_i/Q_i$ such that $f_I \notin (f_i^2)$ to obtain the desired element $f = (0, ..., f_i, ..., 0) \in R$. Now, the connectedness of $\text{Spec}R$ it is equivalent to saying that the only idempotents of R are trivial ones, namely, zero and one.

Lastly, choose $f \in R$ neither a unit nor idempotent. Then, $f \notin f^2$. In fact, by contradiction, if $f = cf^2$, for some $c \in R$, and so $cf(1 - cf) = 0$, which means that cf is idempotent. Hence, $cf = 0$ or $cf = 1$. In the first case we have $f = (cf)f = 0$, and in the second case, f is a unit. Then both cases contradicts our hypothesis on f. ☐

REMARK 2.4. The previous proposition guarantees the existence of non-reduced forcing algebras over any noetherian ring which is not a finite direct product of fields. Specifically, as before, we choose an element $f \in R$, such that $f \notin (f^2)$ and define $A := R/(f^2)$.

Finally, we present a more interesting example of an irreducible but not reduced forcing algebra over an affine domain base ring R such that the codim$((f_1, ..., f_n), R)$ is arbitrary large.

EXAMPLE 2.5. Consider $R = k[x_1, ..., x_{n-1}, z]/(x_1 z, ..., x_{n+1} z)$, $h = x_1 T_1 + \cdots + x_{n+1} T_{n+1} + z^2$ and $A = R[T_1, ..., T_{n+1})/(h)$. Then,

$$\operatorname{codim}((x_1, ..., x_{n+1}), R) = n,$$

because the ring of polynomials is catenary. Besides, it is straightforward to verify that $z \notin (h)$, and $z^3 = z^2 h \in (h)$. Therefore A is non-reduced.

3. Integrity over UFD

The main philosophy developed here is to study very general and rough properties to reach on that process a deeper intuition over the more interesting and fine ones. So, if we demand a forcing algebra being reduced and irreducible, that means exactly demanding integrity. Now, we prove an integrity criterion for forcing algebras over UFD as base ring involving just the height of the forcing elements $f_1, ..., f_n$.

LEMMA 3.1. *Let R be a Noetherian UFD which is not a field, $J = (f_1, \ldots, f_n, f)$, where some $f_i \neq 0$, and let A be the forcing algebra corresponding to this data and $B = R[T_1, ..., T_n]$. Then A is an integral domain if and only if $J = R$, or* $\operatorname{ht} J \geq 2$.

PROOF. Along the proof we will use the basic fact that in a UFD the notions of prime and irreducible element coincide. We will prove the negation of the equivalence $((h) \in \operatorname{Spec} B) \Leftrightarrow (I = R \vee \operatorname{ht} J \geq 2)$, which is equivalent formally to $((h) \notin \operatorname{Spec} B) \Leftrightarrow (I \neq R \wedge \operatorname{ht} J \leq 1)$. Now, we can written the condition at the right side by $\operatorname{ht} J \leq 1$, assuming implicitly that $\operatorname{ht} I$ is well defined, i.e., $I \neq R$. So we will see that A is not an integral domain if and only if $\operatorname{ht} J \leq 1$. In fact, we can assume $J \neq 0$ and therefore $\operatorname{ht} J = 1$. Choose a prime ideal P of R such that P contains J and $\operatorname{ht} P = 1$. Choose $a \neq 0 \in P$. Now, some of the prime factors of a, say p, belongs to P and therefore $P = (p)$, due to the fact that both prime ideals have height one. Thus, there exist $g_i, g \in R$ such that $f_i = pg_i$ and $f = pg$, hence $h = f_1 T_1 + \ldots + f_n T_n + f = p(g_1 T_1 + \ldots + g_n T_n + g)$ is the product of p and an element which is not a unit since some of the f_i is different from zero. Therefore h is not irreducible, which is equivalent of being a non prime element. In conclusion, A is not an integral domain.

Conversely, assume that A is not an integral domain, or equivalently that $h = f_1 T_1 + \ldots + f_n T_n + f$ is not irreducible. Hence, there exists polynomials $Q_1, Q_2 \in R[T_1, \ldots, T_n]$, not units, such that $h = Q_1 Q_2$. Now, the degree of h

is the sum of the degrees of Q_1 and Q_2, because R is an integral domain. Then one of the two factors has degree zero, say Q_1. Comparing the coefficients we get that each $f_i = Q_1 g_i$ and $f = Q_1 g$, and $Q_2 = g_1T_1 + \ldots + g_nT_n + g$. In conclusion, $J \subseteq (Q_1)R$ and therefore by Krull's Theorem $\mathrm{ht}(J) \leq 1$. $\square$

4. A Normality Criterion for Polynomials over a Perfect Field

Now we will try to understand under what conditions on the elements $f_1, \ldots, f_n, f \in R$ the associated forcing algebra is a normal domain in the case that R is the ring of polynomials over a perfect field. For some examples, results and intuition we assume a very basic and modest knowledge of algebraic geometry, mainly relating affine varieties (see, for example [**14**] and [**17**, Chapter I]).

Remark 4.1. If $R = k[x_1, \ldots, x_r]$ and $h = f_1T_1 + \ldots + f_nT_n + f \in B := R[T_1, \ldots, T_n]$, $h \neq 0$, then the forcing algebra $A = R[T_1, \ldots, T_n]/(h)$ is equidimensional of dimension $\dim A = r + n - \mathrm{ht}((h)) = r + n - 1$, since $= R[T_1, \ldots, T_n]$ is catenary and h has pure codimension one, because every minimal prime over (h) has height one by Krull's principal ideal theorem. Therefore in the case that k is a perfect field we deduce from the corollary of the Jacobian criterion (see Chapter 1) that the singular locus of the forcing algebra is exactly the prime spectrum of the following ring

$$A_S = A/((\partial h/\partial x_j), (\partial h/\partial T_i)) = R[T_1, \ldots, T_n]/(h, (\partial h/\partial x_j), (\partial h/\partial T_i)).$$

Now, $(\partial h/\partial x_j) = \sum_{i=1}^{n} (\partial f_i/\partial x_j)T_i + (\partial f/\partial x_j)$ and $\partial h/\partial T_i = f_i$. Thus we get

$$J := (h, (\partial h/\partial x_j), (\partial h/\partial T_i)) = (h, \sum_{i=1}^{n} (\partial f_i/\partial x_j)T_i + (\partial f/\partial x_j), f_i)$$

$$= (f, f_i, \sum_{i=1}^{n} (\partial f_i/\partial x_j)T_i + (\partial f/\partial x_j)),$$

where $i, j \in \{1, ..., n\}$. We can write the last set of generators in a compact way using matrices:

$$\begin{pmatrix} \partial f_1/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \vdots & & \vdots \\ \partial f_1/\partial x_r & \ldots & \partial f_n/\partial x_r \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} + \begin{pmatrix} \partial f/\partial x_1 \\ \vdots \\ \partial f/\partial x_r \end{pmatrix}.$$

We will denote by $\overline{J}$ the class of J in A.

Now we rewrite the normality condition for the forcing algebra A in terms of the codimension of its singular locus $V(\overline{J}) \in \operatorname{Spec} A$, or in terms of the codimension of the corresponding closed subset $V(J) \subseteq \operatorname{Spec}(R[T_1, \ldots, T_n])$, which are isomorphic as affine schemes. On this section we set

$$I = (f, f_1, ..., f_n) \in R$$

and $D = (\partial f/\partial x_i, \partial f_j/\partial x_i)$ for $i, j \in \{1, ..., n\}$. Note that $J \subseteq (I + D)B$. In particular. $V(IB) \cap V(DB) \subseteq V(J) \subseteq \operatorname{Spec} B$.

First, let's consider the trivial case $R = k$. By previous comments we know that if $A \neq 0$ then $A = k[T_1, ..., \check{T}_i, ..., T_n]$, so A is regular and thus a normal domain. In conclusion, A is a normal domain if and only if all f_i and f are zero, or there exists some $f_i \neq 0$.

LEMMA 4.2. *let $R = k[x_1, \ldots, x_r]$ be the ring of polynomials over a perfect field k and $h = f_1T_1 + \ldots + f_nT_n + f \in B := R[T_1, \ldots, T_n]$, with $h \neq 0$, and $A = R[T_1, \ldots, T_n]/(h)$. Then, the following conditions are equivalent:*

(1) A is a normal ring.
(2) $\operatorname{codim}(\overline{J}, A) \geq 2$, or $\overline{J} = A$.
(3) $\operatorname{codim}(J, B) \geq 3$, or $J = B$.

PROOF. (1) $\Rightarrow$ (2) Assume that A is a normal ring, then the Serre's Criterion tells us that for any prime ideal q of A with $\mathrm{ht}q \leq 1$, A_q is a regular ring (remember that in dimension zero regularity is equivalent to being a field). Now, suppose that $\overline{J} \subsetneq A$. Then, we know that for any prime P of A that contains $\overline{J}$, A_P is not regular, therefore $\mathrm{ht}P \geq 2$, thus $\operatorname{codim}(\overline{J}, A) \geq 2$.

(2) $\Rightarrow$ (1) We know that A is C-M because it is the quotient of C-M $R[T_1, \ldots, T_n]$ by an ideal (h) of height one generated by exactly one element, (see [**10**, Theorem 18.13]). Therefore, for any prime ideal P of A the local ring A_P is C-M. Then,

$$\operatorname{depth}(A_P) = \dim(A_P) \geq \min(2, \dim(A_P)).$$

Thus A satisfies the condition (S2) of the Serre's Criterion. Besides, A satisfies condition (R1). In fact, any prime ideal P of A of height at most 1 does not contain $\overline{J}$, because $\operatorname{codim}(\overline{J}, A) = \mathrm{ht}_A(\overline{J}) \geq 2$, or $\overline{J} = A$, hence P is not in the singular locus of A, that means the regularity of the local ring A_P.

Since, $\overline{J} = A$ if and only if $J = B$ then, for the equivalence between (2) and (3) we can assume that $\overline{J} \subsetneq A$ (respectively $J \subsetneq B$).

(2) $\Rightarrow$ (3). Let P a prime ideal of B that contains J, then by hypothesis $\mathrm{ht}_A(\overline{P}) \geq 2$. Let $\overline{P_0} \subsetneq \overline{P_1} \subsetneq \overline{P_2} = \overline{P}$ a chain of primes in A, then one can see the corresponding chain of prime ideals in B adding the zero ideal, which is a prime ideal, $Q_0 = (0) \subsetneq Q_1 = P_0 \subsetneq Q_2 = P_1 \subsetneq Q_3 = P_2 = P$, that means $\operatorname{codim}(J, B) \geq 3$

(3) $\Rightarrow$ (2) Let P be a prime ideal of A that contains $\overline{J}$, and let Q be the prime ideal of B that correspond to P. Clearly, $J \subseteq Q$ as subsets of B. We know that $\mathrm{ht}(Q) \geq 3$ and $(h) \subseteq Q$, therefore Q contains a minimal prime ideal of (h), say Q_0, which has height one by Krull's Principal Ideal Theorem. In

virtue of this, we know that there exists a saturated chain of primes ideals of B,

$$(0) \subsetneqq Q_0 \subsetneqq Q_1 \subsetneqq Q_2 \subseteq Q,$$

since B is a catenary ring and $\mathrm{ht}Q \geq 3$, and therefore any saturated chain of prime ideals from (0) to Q has the same length, that is, $\mathrm{ht}(Q)$, which is a least three. Therefore, looking at the corresponding chain in A, and denoting by P_i the prime ideal of A corresponding to Q_i, we get $P_0 \subsetneqq P_1 \subsetneqq P_2 \subseteq P$, then $\mathrm{ht}P \geq 2$. In conclusion, $\mathrm{codim}(\overline{J}, A) \geq 2$. □

REMARK 4.3. An important fact is that for $R = k[x_1, \ldots, x_r]$, I an ideal of R and $B = R[T_1, ..., T_n]$ we know that the $\mathrm{codim}(I, R) = \mathrm{codim}(IB, B)$, because by previous results we get

$$n + r - \mathrm{codim}(IB, B) = \dim(B/IB) = \dim((R/I)[T_1, \ldots, T_n]) =$$

$$\dim(R/I) + n = \dim R - \mathrm{codim}(I, R) + n = n + r - \mathrm{codim}(I, R).$$

We want to find necessary and sufficient conditions for the forcing data $f_1, ..., f_n$ and f on the base ring of polynomials $R = k[x_1, ..., x_n]$, such that the associated forcing algebra turns out to be a normal domain. The previous lemma gives a condition over A and the Jacobian ideal J of the partial derivatives of the forcing equation, which involves, as seen before, again the forcing ideal and new forcing equations defined by the partial derivatives of the original forcing data. This suggests that a suitable condition for normality over the base R should involve the forcing data and its partial derivatives. The following collection of examples start to give us a good first intuition of the phenomenon.

EXAMPLE 4.4. Let k be a perfect field and let's define $R = k[x, y]$; $B = k[x, y, T_1, T_2]$; $A = B/(h)$ and

$$h = x^a T_1 + y^b T_2 + x^c y^d,$$

where a, b, c and d are nonnegative integers. After computations we have that the Jacobian ideal

$$J = (x^a, y^b, x^c y^d, a x^{a-1} T_1 + c x^{c-1} y^d, b y^{b-1} T_2 + d x^c y^{d-1}).$$

Let $D \subseteq R$ be the ideal generated by all the partial derivatives of the generators of the forcing ideal $I = (f_1, f_2, f) = (x^a, y^b, x^c y^d)$, i.e.,

$$D = (a x^{a-1}, b y^{b-1}, c x^{c-1} y^d, d x^c y^{d-1}).$$

By Lemma 3.1, A is a domain for any nonnegative values of the exponents.

After elementary considerations we see that $\mathrm{codim}(J, B) \geq 3$ or $J = B$ if and only if some of the following seven cases occur:

i) $a = 0$.

ii) $a = 1$.

iii) $b = 0$.

iv) $b = 1$.
v) $d = c = 0$.
vi) $c = 1$ and $d = 0$.
vii) $c = 0$ and $d = 1$.

In fact, in any other case $J \subseteq (x, y)B$, and therefore $\text{codim}(J, B) \leq 2$. Moreover, it is also elementary to see that the previous seven cases are exactly the ones in which the ideal $I + D$ is equal to R.

In conclusion, in virtue of the previous Lemma, A is a normal domain if and only if $I + D = R$.

REMARK 4.5. Suppose that k is an algebraically closed field. Continuing with the notation of the former example, let's write $V = V(I) \subseteq k^2$, $W = V(D) \subseteq k^2$, $Y = V(h) \subseteq k^4$ and $S = V(J) \subseteq k^4$ denote the corresponding affine varieties and $\pi : S \to V$ the natural projection to the first two coordinates. Geometrically, Example 4.4 suggests that the normality of the variety X (which is equivalent to the normality of the forcing algebra, see [**17**, Exercise I.3.17]), is related to the intersection of V and W, because $V \cap W = \emptyset$, if and only if $I + D = R$. In fact, this is true for arbitrary polynomial data f_1, f_2 and $f \in R$ as we will see.

First, by Lemma 3.1, A is an integral domain if and only if $\text{ht}I \geq 2$ or $I = R$. So, let's assume that A is a domain and $I \subsetneq R$, otherwise $V = \emptyset$ and $J = B$, being A normal, by Lemma 4.2. Thus, $\text{ht}I \geq 2$, which means that the minimal prime ideals over I are just finitely many maximal ideals, since $\dim R = 2$. But, by the Nullstellensatz (see [**1**, Exercise 7.14]) this points correspond exactly to the points of V. Therefore, let's write $V = \{v_1, ..., v_r\}$.

Moreover, let S be the singular locus of Y in the sense that, if we consider S as a subvariety of Y. By previous comments S is the finite union of its (singular) fiber varieties $S_{v_i} = \pi^{-1}(v_i)$. Now, by Lemma 4.2, Y is a normal variety if and only if $\text{codim}(S, K^4) \geq 3$ (which is equivalent to $\text{codim}(S, Y) \geq 2$).

Assume, that $V \cap W \neq \emptyset$, i.e., $I + D \subsetneq R$, and let's prove that Y is not normal. In fact, we know that $J \subseteq (I + D)B$. Therefore, by Remark 4.3

$$\text{codim}(S, k^4) = \text{codim}(J, B) \leq \text{codim}((I + D)B, B) = \text{codim}(I + D, R) \leq 2,$$

implying that Y is not normal.

Conversely, assume that $V \cap W = \emptyset$. Then, for any point $v \in V$, there exists some $\partial f_i(v)/\partial x_j \neq 0$, because if not all the partial derivatives of the forcing data would be zero at v (the elements $\partial f(v)/\partial x_j$ are also zero, because we can write them as a linear combinations of the $\partial f_i(v)/\partial x_j$, see Remark 4.1), implying that $v \in W$, but that is impossible.

Clearly, $S_v = V(G)$, where $v = (a, b) \in k^2$ and

$$G = (x - a, y - b, \partial f_1(v)/\partial x T_1 + \partial f_2(v)/\partial x T_2 + \partial f(v)/\partial x,$$

$$\partial f_1(v)/\partial y T_1 + \partial f_2(v)/\partial y T_2 + \partial f(v)/\partial y).$$

But, under the condition that some $\partial f_i(v)/\partial x_j \neq 0$, it is elementary to see that $\operatorname{codim}(G, B) \geq 3$. In conclusion, $\operatorname{codim}(S_v, k^4) \geq 3$, implying that $\operatorname{codim}(S, k^4)$, being the minimum of the codimension of its singular fibers, is bigger or equal than three, which means the normality of Y.

Besides, if we move to the next dimension, i.e., $R = k[x_1, x_2, x_3]$ and $B = R[T_1, T_2, T_3]$, then, it is possible to see in a natural way that a necessary condition for the normality of Y is that $(\dim V \cap W) < 1$ (here we assume that the dimension of the empty set is -1). Because, suppose by contradiction that $\dim V \cap W \geq 1$. For any point $v \in V \cap W$, by Remark 4.1 and Lemma 3.1 (Ch. 1) the fiber $S_v \cong \mathbb{A}^3_k$. Therefore, $(V \cap W) \times \mathbb{A}^3_k \subseteq S$. But, $\dim(V \cap W) \times \mathbb{A}^3_k \geq 1 + 3 = 4$, and so, $\dim S \geq 4$, thus, $\operatorname{codim}(S, k^6) \leq 2$, implying that Y is not normal. Note that this argument works independent from the number of variables. However, this case was very suitable to obtain the right intuition about the desired condition i.e., $\dim(V \cap W) < r - 2$.

Heuristically, one can compute the dimension of S by knowing the general behavior of the dimension of the fibers S_v and the dimension of the base space V. Now, by Lemma 3.1 (Ch. 1) the fibers S_v have maximal dimension exactly when the rank of the forcing matrix is minimal, i.e., when the point v belongs to $W \cap V$. Therefore, to guarantee that the dimension of Y is not so big (in order to maintain the codimension big enough), we need to bound the dimension of the subvariety of V with maximal dimensional singular fibers, i.e., the dimension of $V \cap W$. In fact, assuming that Y is irreducible, the right necessary and sufficient condition for Y being an (irreducible) normal variety is that ($\dim V \leq r - 2$ and) $\dim V \cap W \leq r - 3$, where $V, W \subseteq k^r$.

First, in order to get a better intuition about the fibers, the following proposition tells us that the points of $\operatorname{Spec} R$ with fibers completely singular are exactly the points of $V(I) \cap V(D)$.

PROPOSITION 4.6. *Let $R = k[x_1, \ldots, x_r]$ be the ring of polynomials over a perfect field k; $B = R[T_1, \ldots, T_n]$; $h = f_1 T_1 + \cdots + f_n T_n + f$; $f, f_1, \ldots, f_n \in R$; $A = B/(h)$; $I = (f, f_1, \ldots, f_n)$; $D = (\partial f/\partial x_j, \partial f_i/\partial x_j)$ and*

$$J := (h, (\partial h/\partial x_j), (\partial h/\partial T_i)).$$

Let $\varphi : Y = \operatorname{Spec} A \to X = \operatorname{Spec} R$ be the forcing morphism. Choose a point $x \in Y$ with nonempty fiber $\varphi^{-1}(x)$. Then $x \in X$ has fiber completely singular i.e., $\varphi^{-1}(x) \subseteq V(J) \in Y$ if and only if $x \in V(I + D) \subseteq X$.

PROOF. We know from the Corollary of the Jacobian Criterion that for any prime ideal $y \in Y$, A_y is not regular if and only if $y \in V(J)$. Let $x \in V(I + D)$ and $Q \in \varphi^{-1}(x)$. Then, $(I + D)B \in Q$ and so $J \in Q$, meaning that $Q \in V(J)$.

Conversely, let's consider a point $x \in X$, such that $\varphi^{-1}(x) \subseteq V(J)$. Now, it is elementary to see that the last condition means that $\varphi^{-1}(x) = V(J_x)$, where

$$\varphi^{-1}(x) = A = k(x)[T_1, \ldots, T_n]/(f_1(x)T_1 + \ldots + f_n(x)T_n + f(x)),$$

and $J_x = (\sum_{i=1}^{n}(\partial f_i(x)/\partial x_j)T_i + (\partial f(x)/\partial x_j)$, for $i, j \in \{1, ..., n\}$.

Firstly, if $f_i \notin x$, for some i, then the fiber $\varphi^{-1}(x)$ is completely regular, because, by previous comments (Ch. 1 §2) $\varphi^{-1}(x) \cong \mathbb{A}^{n-1}_{k(x)}$.

Secondly, if $f \notin x$, then $f(x) = 0$. But, we know that $f_1(x) = \cdots = f_n(x) = 0$, therefore the fiber is empty, since $h = f(x) \neq 0 \in k(x)$. But, it contradicts our hypothesis. Note that, until now, we know that $h = f_1(x)T_1 + \ldots + f_n(x)T_n + f(x) = 0$.

Thirdly, suppose that $\partial f_i/\partial f_j \notin x$, for some $i, j \in \{1, ..., n\}$, that means, $\partial f_i(x)/\partial f_j = 0$. We consider two cases: Suppose that $\partial f(x)/\partial f_j \neq 0$. Then, since $h = 0$, the ideal $Q = (T_1, ..., T_n) \in \varphi^{-1}(x)$, but

$$\sum_{i=1}^{n}(\partial f_i(x)/\partial x_j)T_i + (\partial f(x)/\partial x_j) \notin Q.$$

Therefore $Q \notin V(J_x)$, a contradiction. In the second case, i.e., $\partial f(x)/\partial f_j = 0$,, the prime ideal $Q' = (T_1, ..., T_i - 1, ..., T_n) \in \varphi^{-1}(x)$, but

$$\sum_{i=1}^{n}(\partial f_i(x)/\partial x_j)T_i + (\partial f(x)/\partial x_j) =$$

$$\sum_{i=1}^{n}(\partial f_i(x)/\partial x_j)T_i \notin Q'.$$

So, again, $Q' \notin V(J_x)$, a contradiction.

Lastly, if $\partial f(x)/\partial x_j \neq 0$, for some j, then, due to the last results

$$\sum_{i=1}^{n}(\partial f_i(x)/\partial x_j)T_i + (\partial f(x)/\partial x_j) = \partial f(x)/\partial x_j \in J_x,$$

thus $\varphi^{-1}(x) = V(J_x) = \emptyset$. But, this is not possible, because the fiber is not empty.

In conclusion, $\varphi^{-1}(x) \subseteq V(J) \in Y$, as desired. □

Now, we present the statement of the normality criterion for forcing algebras over the ring of polynomials with coefficients in a perfect field.

THEOREM 4.7. *Let $R = k[x_1, \ldots, x_r]$ be the ring of polynomials over a perfect field k; $B = R[T_1, \ldots, T_n]$; $f, f_1, \ldots, f_n \in R$; $I = (f, f_1, \ldots, f_n)$;$D = (\partial f/\partial x_j, \partial f_i/\partial x_j)$, for $i, j \in \{1, ..., n\}$. Then, the forcing algebra for this data A is a normal domain if and only if the following two conditions hold:*

(a) $\operatorname{codim}(I, R) \geq 2$*, or* $I = R$*.*

(b) $\operatorname{codim}(I+D,R)>2$*, or* $I+D=R$.

Moreover, in the case that all $f_i=0$*, then (b) is a necessary and sufficient condition for* A *being a normal ring.*

Proof. We have already proved in Lemma 3.1 that (a) is a necessary and sufficient condition for A being an integral domain. Let's prove that (b) is equivalent to normality. Effectively, following Lemma 4.2 we just need to see the condition (b) is equivalent to $\operatorname{codim}(J,B)>2$, or $J=B$. Let's denote the last condition by (b'). By Remark 4.1 we know that $J\subseteq(I+D)B$. Suppose that (b') holds. First, if $J=B$, then $(I+D)B=B$, implying $I+D=R$. Second, if $\operatorname{codim}(J,B)>2$, then by Remark 4.3 we get

$$\operatorname{codim}(I+D,R)=\operatorname{codim}((I+D)B,B\geq\operatorname{codim}(J,B)>2.$$

Conversely, assume that (b) holds and $J\neq B$. We prove that $\operatorname{codim}=(J,B)>2$. Let Q be a prime ideal of B that contains J. First, assume that $(I+D)B\subseteq Q$, then $I+D\neq R$, therefore $\operatorname{codim}(I+D,R)>2$, so, again by Remark 4.3 $\operatorname{codim}((I+D)B,B)>2$, which implies that $\operatorname{codim}(Q,B)>2$. Second, suppose that $I+D\nsubseteq Q$, then necessarily one of the partial derivatives $\partial f/\partial x_j$ or $\partial f_i/\partial x_j$ is not contained in Q, because $IB\subseteq J\subseteq Q$. In fact, there exits some $b\in 1,\ldots,n$ and some $c\in 1,\ldots,r$ with $\partial f_b/\partial x_d\notin Q$, cause if not, all $\partial f_i/\partial x_j$ would be contained in Q and also the elements $\sum_{i=1}^{n}(\partial f_i/\partial x_j)T_i+\partial f/\partial x_j$ and therefore $\partial f/\partial x_j$ for any j, thus D would be also contained in J, which is not the case. For simplicity suppose that Q not contained the element $\alpha:=\partial f_1/\partial x_1$ and let's write $l:=\sum_{i=1}^{n}(\partial f_i/\partial x_1)T_i+\partial f/\partial x_1$. Let ψ be the following homomorphism of $R_{(\alpha)}$ algebras

$$\psi: B_{(\alpha)}\approxeq R_{(\alpha)}[T_1,\ldots,T_n]\longrightarrow R_{(\alpha)}[T_2,\ldots,T_n],$$

that sends T_1 to $g:=-\alpha^{-1}(\sum_{i=2}^{n}(\partial f_i/\partial x_1)+\partial f/\partial x_1)$ and T_j to T_j, for $j\geq 2$. Clearly, ψ is surjective. Moreover, $ker(\psi)=(T_1-g)$. To see this let $S\in ker(\psi)$. Then using the binomial expansion we can write it in the form:

$$\begin{aligned}S=S(x_1,\ldots,x_r,(T_1-g)+g,\ldots,T_n)&=S_0(x_1,\ldots,x_r,(T_1-g),\ldots,T_n)+\\&\quad S(x_1,\ldots,x_r,g,\ldots,T_n),\\&=S_0(x_1,\ldots,x_r,(T_1-g),\ldots,T_n)+\psi(S)\\&=S_0(x_1,\ldots,x_r,(T_1-g),\ldots,T_n),\end{aligned}$$

with S_0 being divisible by T_1, which implies that the former expression is divisible by T_1-g. Thus $S\in(T_1-g)$.

On the other hand, in the ring $R_{(\alpha)}[T_1,\ldots,T_n]$ we know that $(T_1-g)=(l)$, therefore ψ induces an isomorphism between $R_{(\alpha)}[T_1,\ldots,T_n]/(l)$ and $R_{(\alpha)}[T_2,\ldots,T_n]$. Denote by Q_0 the image under ψ of $QR_{(\alpha)}[T_1,\ldots,T_n]$, and assume for the sake of contradiction that $\operatorname{codim}(Q,B)\leq 2$ then we have the following chain of inequalities:

$$d:=\dim(B/Q)=\dim B-\operatorname{codim}(Q,B)=n+r-\operatorname{codim}(Q,B)\geq n+r-2.$$

Besides, B is a Jacobson ring, hence there exists a maximal ideal m containing Q such $\alpha \notin m$, otherwise α would be contained in the intersection of all the maximal ideals containing Q, which is Q, absurd. Now, let's consider a saturated chain of primes ideals from Q to m, which exits in virtue of Zorn's lemma. Besides, this chain has length exactly d because B/Q is an affine domain and therefore, d is the length of any saturated chain of primes on it (see fundamental results on Chapter 1). Then,

$$Q = Q_0 \subsetneqq Q_1 \subsetneqq, \dots, \subsetneqq Q_{d-1} \subsetneqq Q_d = m.$$

Now, we can consider this chain in $R_{(\alpha)}[T_1, \dots, T_n]$, because no Q_i contains α. This shows that $\dim(R_{(\alpha)}[T_1, \dots, T_n])/Q^e \geq d$ and, in fact, the equality holds because we are localizing and thus the dimension cannot be bigger that the dimension of the original ring. Besides, ψ induces an isomorphism between $R_{(\alpha)}[T_1, \dots, T_n]/Q^e$ and $R_{(\alpha)}[T_2, \dots, T_n]/Q_0$, then finally, recalling that $\operatorname{codim}(I, R) \geq 2$ and that $l \in Q$ we get

$$\begin{gathered} d = \dim(R_{(\alpha)}[T_1, \dots, T_n])/Q^e) = \dim(R_{(\alpha)}[T_2, \dots, T_n])/Q_0) \leq \\ \dim(R_{(\alpha)}[T_2, \dots, T_n])/I^e) \leq \dim(R[T_2, \dots, T_n])/I^e) \\ = \dim((R/I)[T_2, \dots, T_n]) = \dim(R/I) + n - 1 = \\ \dim R - \operatorname{codim}(I, R) + n - 1 \leq r + n - 1 - 2 < n + r - 2. \end{gathered}$$

Which is a contradiction with the former estimate of d. Finally, if all $f_i = 0$ then $J = I + D$ and then from the fact that $\operatorname{codim}((I + D), R) = \operatorname{codim}((I + D), B)$ we deduce from Lemma 4.2 that condition (b) is equivalent to the normality of A.

□

Now, we state a direct application of the previous Theorem to normal affine varieties. As said before, our convention is that $\dim \emptyset = -1$.

COROLLARY 4.8. *Let $R = k[x_1, \dots, x_r]$ be the ring of polynomials over an algebraically closed field k; $B = R[T_1, \dots, T_n]$; $f, f_1, \dots, f_n \in R$; $I = (f, f_1, \dots, f_n)$;$D = (\partial f/\partial x_j, \partial f_i/\partial x_j)$. Assume that (h) is a radical ideal, where $h = f_1T_1 + \cdots + f_nT_n + f$. Let's denote by $V = V(I) \subseteq k^r$ and $W = V(D) \subseteq k^r$ the affine varieties defined by I and D, respectively. Then, $X = V(H) \subseteq k^{n+r}$ is a normal (irreducible) variety if and only if the following two conditions holds simultaneously*

(1) $\dim V \leq r - 2$.
(2) $\dim(V \cap W) < r - 2$.

Moreover, in the case that all $f_i = 0$, then (2) is a necessary and sufficient condition for X being a normal (irreducible) variety.

PROOF. Recall that a variety is normal if for any point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a normal domain (see [**17**, Exercise I.3.17]). Since (h) is a radical ideal, we know that the forcing algebra $A = B/(h)$ is exactly the ring of coordinates of X. Since X is affine and normality is a local property we have that X is a normal (irreducible) variety if and only if A is a normal domain. Besides, from Hilbert's Nullstellensatz we get

$$\dim V = \dim(R/I(V)) = \dim(R/\operatorname{rad}(I)) = \dim(R/I)$$

$$= \dim R - \operatorname{codim}(I, R) = r - \operatorname{codim}(I, R),$$

and analogously

$$\dim(V \cap W) = r - \operatorname{codim}(I + D, R).$$

From this and the fact that $V = \emptyset$ (or $V \cap W = \emptyset$), if and only if $I = R$ (or $I + D = B$), we rewrite the conditions (a) and (b) of the former theorem as (1) and (2). □

As a comment, we say that the discussion beginning at Example 4.4 is essentially the way in which the above criterion of normality was discovered. However, the formal proof that we present, do not give explicitly more intuition and understanding of the phenomenon that the discussion above. In fact, we can say informally, that the "right proof" was mainly a suitable collection of examples, in which we increase our intuition and generality step by step. Moreover, the final proof was, in some sense, a "natural" consequence of the intuition that we got by means of the examples.

Moreover, and moving us for a while into the philosophical understanding of mathematics, this criterion is an example of a theorem of mathematics, who was discovered by considering a "right" sequence of examples, more than doing formal and abstract considerations. It suggests the possibility of exploring a "theory of examples" in mathematics, in which we study the examples and sequences of examples as formal objects, and when these sequences of examples "converge" to "more general" examples (i.e., theorems). And conversely, how an specific example can be "approximate" by a sequence of other ones. This could give deep understanding of how the mathematical discovery process works beyond the mathematical formalism.

Lastly, in order to support the former intuition we dedicate the next pair of sections to study two interesting and enlightening examples.

5. An Enlightening Example

In this section we study an specific example of a forcing algebra with several forcing equations and we explore the properties studied until now. This example shows how rich and interesting could be the formal study of forcing algebras on its own.

Let $R = k[x, y]$ be the ring of polynomials over a (perfect) field k, $B = R[T_1, T_2]$, $A = B/H$, where

$$H = (h_1, h_2) = (xT_1 + yT_2, yT_1 + xT_2) = \left(\begin{pmatrix} x & y \\ y & x \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \right).$$

The determinant of the associated matrix M is $x^2 - y^2 = (x+y)(x-y)$. It is easy to check that h_1 is irreducible and that h_2 does not belong to the ideal generated by h_1. Therefore $h_1, h_2 \subseteq B$ is a regular sequence and hence, by Proposition 3.3 H has pure codimension 2.

Let P be a minimal prime of H. Then, by a previous remark, P contains the elements $\det MT_i = (x-y)(x+y)T_i$ for $i = 1, 2$. If $\det M \notin P$, then $T_i \in P$, and therefore $P = (T_1, T_2)$. Now, assume that $\det M \in P$, then $x - y \in P$ or $x + y \in P$. In the first case, $h_1 - T_1(x-y) = y(T_1 + T_2)$ should be in P. But, if $y \in P$ then $x = (x-y) + y \in P$, which implies that $P = (x, y)$. If $T_1 + T_2 \in P$ then it is easy to check that $P = (x - y, T_1 + T_2)$, since this is a prime ideal containing H. On the other hand, if $x+y \in P$, then, similarly we see that $P = (x, y)$, or $P = (x + y, T_1 - T_2)$. In conclusion, the minimal primes of H (which are, in fact, the associated primes of H, because A is a Cohen-Macaulay ring) are the four ideals $P_1 = (T_1, T_2)$, $P_2 = (x, y)$, $P_3 = (x - y, T_1 + T_2)$ and $P_4 = (x + y, T_1 - T_2)$.

This example shows that Theorem 1.1 is false for several forcing equations, since $\operatorname{Spec} A$ is not an irreducible space but the ideal generated by the forcing data (x, y) has height two.

Let $V_i = V(P_i) \subseteq k^4$ be the affine variety define by P_i, which correspond to the irreducible components of $V = V(H)$. Now, the intersections of any couple of this components correspond to singular points of V (we assume for a while that k is algebraically closed, and we replace H by $\operatorname{rad} H$ in order to work with the corresponding variety V), because the ring of coordinates of V localized at the maximal ideal corresponding to such a points has at least two irreducible components and therefore it is not an integral domain, in particular, it is not a regular local ring, since local regular rings are domains.

This is a way to see geometrically the non-normality of V, because the normality is a local property and the localization at these intersection points, say $\mathfrak{p} \in \operatorname{Spm}(A)$, is not a normal ring. In fact, a local ring has, clerarly, connected spectrum, therefore $A_{\mathfrak{p}}$ cannot be a direct product of normal domains (Ch. 2 §1). Besides, by the former comment, $A_{\mathfrak{p}}$ cannot be neither a normal domain.

Returning to our computations, we see that the intersection of these irreducible components are, in general, defined by lines and, in two cases, defined by just one point. In fact, $V_1 \cap V_2 = V(x, y, T_1, T_2)$; $V_1 \cap V_3 = V(T_1, T_2, x - y)$; $V_1 \cap V_4 = V(T_1, T_2, x+y)$; $V_2 \cap V_3 = V(x, y, T_1 + T_2)$; $V_2 \cap V_4 = V(x, y, T_1 - T_2)$ and $V_3 \cap V_4 = V(x, y, T_1, T_2)$.

Furthermore, by Proposition 1.1 (Ch.2), $\operatorname{Spec} A$ is connected, since we are in the homogeneous case. Moreover, in relation with Lemma 2.1 (Ch.2),

$V(P_1)$ is an horizontal component, $V(P_2)$ a vertical component and $V(P_3)$ and $V(P_4)$ behave like "mixed" components i.e., they do not dominate the base nor are they the preimage of a subset of the base. Besides, $\operatorname{Spec} A$ is also locally (over the base) connected because every pair of minimal components have non-empty intersection and the elementary fact that the minimal primes of a localization are exactly the minimal primes of the original ring not intersecting the multiplicative system.

In the case that k is a perfect field, we can use also the Jacobian Criterion in order to prove again tha A is not a normal ring. In fact, by Proposition 3.3, the pure codimension of H is two, since $\{h_1, h_2\}$ is a regular sequence. So, the singular locus in $\operatorname{Spec} A$ is given by the 2×2 minors of the Jacobian matrix defined by the partial derivatives of the h_i, that is,

$$J = \left(T_1^2 - T_1^2, x^2 - y^2, yT_1 - xT_2, xT_1 - yT_2\right).$$

Thus in order to test normality we should find the codimension of J in A and determine if it is bigger or equal than two. Since the pure codimension of H is two we can translate our problem to the ring of polynomial in four variables $B = k[x, y, T_1, T_2]$ and to test if the corresponding Jacobian ideal

$$J_0 = \left(T_1^2 - T_1^2, x^2 - y^2, yT_1 - xT_2, xT_1 - yT_2, h_1, h_2\right)$$

has codimension bigger or equal to four (in general, the codimension of a prime ideal decreases in n, if we mod out by ideals of pure codimension n, mainly because an affine domain is catenary and its dimension is the length of any maximal chain of prime ideals). But, after some computations we can show that the prime ideals that contain J_0 are exactly the ideals defining the varieties corresponding to the intersections of pairs of the irreducibles components of V. That is, (x, y, T_1, T_2), $(T_1, T_2, x - y)$, $(T_1, T_2, x + y)$, $(x, y, T_1 + T_2)$ and $(x, y, T_1 - T_2)$. Therefore $\operatorname{codim}(J_0, B) = 3$, and then, $\operatorname{codim}(\overline{J_0}, A) = 1 < 2$, implying that A does not satisties Serre's condition (R1). Hence, by Serre's Normality Criterion A is not a normal ring. Moreover, by the same reason $B/\operatorname{rad} H$ is not a normal ring, and this is equivalent to the non-normality of the variety $V(H) \subseteq k^4$.

Geometrically, if k is an algebraic closed field, it means just that the singular points of V, which correspond to the maximal ideals containing J_0, are exactly the points in the intersections of the different irreducible components of the variety, which correspond to the geometrical intuition of singularities.

This example suggests the following conjecture.

Conjecture 5.1. In the homogeneous case, assume that $R = k[x_1, ..., x_r]$, and suppose $H = (h_1, ..., h_m) = P_1 \cap ... \cap P_s$, where P_i are the minimal primes, for $i = 1, ..., s$. Then $V(P_i) \cap V(T_1, ..., T_n) \neq \emptyset$.

6. An Example of Normalization

On this section we will compute explicitly the normalization of a forcing algebra by elementary methods illustrating how good examples lead us in a natural way to the study of general basic properties of normal domains.

Let k be a perfect field. Our example is a particular case of the Example 4.4. Let $R = k[x, y]$, $B = R[t, s]$, $A = B/(h)$, where $h = x^2t + y^2s + xy$. Now, with the notation of section 3, $I = (x^2, y^2, xy)$, $D = (x, y)$, and so, $I + D = (x, y)$. By Theorem 4.7 A is a non-normal domain, because $\text{codim}(I, B) \geq 2$, but $\text{codim}(I + D, B) = 2$. Besides, the integral closure, or normalization of A, $\overline{A}$ is a module-finite extension of A (in general, this is true for finitely generated algebras over complete local rings, see [**27**, Exercise 9.8]).

Now, we will give an explicit description of $\overline{A}$ as an affine domain.

First, let $K = K(A)$ be the field of fractions of A and let $u = tx/y \in K$. Then, if we consider the forcing equation h in $K[t, s]$, we get the following integral equation for u, after multiplication by t/y^2:

$$(tx/y)^2 + (tx/y) + st = 0.$$

Let $A' = A[u]$ be the $A-$subalgebra of K generated by u. So, we rewrite h considered in A', by means of $yu = xt$, to obtain the equation $0 = h = y(xu + ys + x)$. But, $y \neq 0$, therefore $xu + ys + x = 0$.

Let $C = k[X, Y, T, S, U]$ be the ring of polynomials. Define $\phi : C \to A'$ the homomorphism of $k-$algebras sending each capital variable into its corresponding small variable. Note that from the previous considerations the ideal $P = (YU - XT, XU + YS + X, U^2 + U + TS) \subseteq \ker \phi$. We will see that $P = \ker \phi$. Effectively, let's write $E = k[X, Y, U, T]/(YU - XT)$. Then, E is a forcing algebra and by Theorem 4.7 is a normal domain.

First, we prove that P is a prime ideal. Define $Q = K(E)$, then, informally if we consider the equations

$$XU + YS + X = U^2 + U + TS = 0$$

in the variable S and solve them, it lead us to obtain the equality $S = -(U^2 + U)/T = -(XU + X)/Y$ in a "suitable" field of fractions. But, in fact, it hods that

$$-(U^2 + U)/T = -(XU + X)/Y \in Q,$$

because

$$-Y(U^2 + U) = -TXU - XT = -T(XU + X) \in D$$

, due to the fact that $YU = XT \in E$. Write $S' = -(U^2 + U)/T = -(XU + X)/Y \in Q$ and consider the natural homomorphism $\psi : E[S] \to E[S'] \subseteq Q$, where $E[S]$ denote the ring of polynomials in the variable S. We will prove that $\ker \psi = (XU + YS + X, U^2 + U + TS)$. For that we need the following basic lemma about normal domains:

LEMMA 6.1. *Let R be a normal domain, $q \in K(R)$,*

$$I = (bx - a \in R[x] : q = a/b; a, b \in R\},$$

and $(R : q) = \{b \in R : bq \in R\}$ be the denominator ideal. Consider the homomorphism of $R-$algebras

$$\varphi : R[x] \to R[q] \subseteq K(R),$$

sending x to q. Then the following holds:

(1) *If $q \notin R$, then* $\operatorname{codim}((R : q), R) = 1$*.*
(2) *Suppose that $(R : q) = (b_1, ..., b_m) \in R$, such that $q = a_i/b_i$, for some $a_i \in R$. Then, $I = (b_1x - a_1, ..., b_mx - a_m)$.*
(3) $\ker \varphi = I$*.*

PROOF. (1) It is a well known fact that any normal Noetherian domain is the intersection of its localizations on primes of height one (see [**10**, Corollary 11.4]). We argue by contradiction. If $\operatorname{codim}((R : q), R) \geq 2$, then $(R : q)$ is not contained in any prime ideal $P \subseteq R$ of height one. In particular, there exists for every such prime ideal P an element $b_P \notin P$, but $b_P \in (R : q)$, meaning that there is $a_P \in R$, with $q = a_P/b_P \in R_P$. In conclusion, $q \in \cap_{\operatorname{ht} P=1} R_P = R$.

(2) Let $bx - a \in I$. That means, in particular, that $b \in (R : q)$. So, we can write $b = c_1b_1 + \cdots + c_rb_r \in R$, for some $c_i \in R$, $i = 1, ..., r$. Now, let $a_i \in R$ be elements such that $q = a_i/b_i$. Since,

$$a = bq = \sum_{i=1}^{n} c_ib_iq = \sum_{i=1}^{n} c_ia_i,$$

it is straightforward to verify $bx - a = \sum_{i=1}^{r} c_i(b_ix - a_i)$, as desired.

(3) Clearly $I \subseteq \ker \varphi$. For the other containment, let $f \in \ker \varphi$ we argue by induction on the degree of f. Write $f = v_nx^n + \cdots + v_0$. The case $n = 1$ is clear. So, assume $n \geq 2$. First, we know that

$$v_nq^n + \cdots + v_0 = 0 \in K(R),$$

then after multiplying by v_n^{n-1}, we get the integrity equation for v_nq,

$$(v_nq)^n + v_{n-1}v_n(v_nq)^{n-1} + \cdots + v_0v_n^{n-1} = 0.$$

So, $v_nq \in R$, because R is a normal domain. Therefore, there exists $d \in R$ such that $q = d/v_n$. Now, $f - x^{n-1}(v_nx - d) \in \ker \varphi$, and it has lower degree. Thus, by the induction hypothesis $f - x^{n-1}(v_nx - d) \in I$, and then $f \in I$, because $v_nx - d \in I$. □

We continue with our discussion, by abuse of notation we write with the same capital letters its classes in E. Now, we know that $Y, T \in (E : S')$. Besides, $(X, T) \in E$ in a prime ideal of codimension one in E, therefore in virtue of Lemma 6.1(1), $(Y, T) = (E : S')$. Hence, applying again Lemma 6.1(2)-(3) we see that

$$\ker \psi = ((Y)S + (XU + X), (T)S + (U^2 + U)),$$

as desired. In conclusion,

$$E[S]/(XU + YS + X, U^2 + U + TS) \cong E[S']$$

is an integral domain, therefore

$$C/P \cong E[S]/(XU + YS + X, U^2 + U + TS)$$

so is.

On the other hand, since the extension $A \to A'$ is integral, both rings have the same dimension (it is a direct consequence from the Going Up, see [**10**, Proposition 4.15]). But, $\dim A = \dim B - \mathrm{ht}(h) = 3$, and then

$$3 = \dim A' = \dim C/\ker \phi = 5 - \mathrm{ht}(\ker \phi),$$

implying $\mathrm{ht}(\ker \phi) = 2$. Besides, it is easy to check that $P \subseteq \ker \phi$ is a (prime) ideal of height strictly bigger that one, therefore both ideals coincide. Finally, we can apply Corollary 1.3 to the affine domain C/P. After computations we verify that

$$(U + 1)(2U + 1), U(2U + 1), U(U + 1) + ST, ST \in J,$$

where J denotes the Jacobian ideal, defining the singular locus of C/P. But, easily we check that

$$C = ((U + 1)(2U + 1), U(2U + 1), U(U + 1) + ST, ST),$$

therefore the singular locus is empty and then C/P is regular, and in particular, normal. In conclusion, an explicit description of the normalization of A as an affine ring is

$$\overline{A} \cong k[X, Y, T, S, U]/(YU - XT, XU + YS + X, U^2 + U + TS).$$

Remark 6.2. A next natural step would be to compute the normalization for forcing algebras with forcing equations of the form $h = x^n t + y^n + xy$, for $n \geq 2$. However, just for the case $n = 3$, new methods seem to be needed. In particular, we get an ideal

$$P = (YU - X^2T, XU + X + Y^2S, U^2 + U + XYST).$$

But, in order to apply Lemma 6.1, the most challenging part appears to be finding an explicit description of the generators of the corresponding denominator ideal, cause

$$S' = -(X + UX)/Y^2 = -(U^2 + U)/XYT,$$

and therefore we just know that $Y^2, XYT \in (D : S')$, where

$$E = k[X, Y, U, T]/(YU - X^2T).$$

But, on this case the ideal (Y^2, XYT) is not prime as in the argument before where we get the prime ideal (X, Y) as denominator ideal.

Finally, we mention that this section suggests on its own a humble way for forthcoming research on computing the normalization of forcing algebras.

CHAPTER 4

The Socle-Parameters Conjecture

b

1. Introduction

The results and methods that we present in this chapter have their origins in the work of J. D. Vélez. First, the idea of proving the DSC by means of annihilators started in his thesis (see [**42**, Lemma 3.1.2.] and [**40**]). The reduction to the case where $S = T/J$, and T is a Gorenstein local ring and J is a principal ideal was stated in a private communication from J. D. Vélez to M. Hochster in 1996, and appears more explicitly in [**41**], and in the Master's Thesis of L. Junes (see [**28**]). Independently, similar results were obtained by J. Strooker and J. Stückrad (see [**39**]).

Besides, the new aspects our results are essentially the following:

Firstly, we present an equivalent form of the DSC in terms of an estimate of the difference of the lengths of the two first Koszul homology groups of quotients of Gorenstein rings by principal zero-divisor ideals. This approach was obtained by us as a natural consequence of J. D. Vélez' former work and as the former series of references show. There is a similar result, obtained independently, although in a rather different context i.e., for homomorphic images of unramified equicharacteristic regular local rings by complete intersection and almost complete intersection ideals. due to S.P. Dutta and P. Griffith (see [**9**, Theorem 1.5]).

Secondly, this new equivalent form of the DCS is presented in two different, but equivalent ways: the former one concerning homological estimates of Gorenstein local rings and a second one, and entirely new form, involving a condition in terms of liftings of socle elements and zero divisors on Gorenstein local rings (see section 6). This condition helps us to perform more explicitly elementary computations (see proof of Proposition 7.1).

2. Preliminary results

Let $(R, m) \hookrightarrow S$ be a module finite extension. Let $s_1, ..., s_n \in S$ be generators of S as an $R-$module. Then, there exist monic polynomials $f_i(y_i) \in R[y_i]$ such that $f_i(y_i) = y_i^{m_i} + a_{i1}y_i^{n_i-1} + \cdots + a_{im_i}$, $a_{ij} \in R$ and $f_i(y_i) = 0$. One can define a homomorphism of $R-$algebras from

$$T := R[y_1, ..., y_n]/(f_1(y_1), ..., f_n(y_n))$$

to S sending each y_i to s_i. If J denotes the kernel of this map, $S \cong T/J$. Finally, since $\dim T = \dim R = \dim T/J$, by reason of the finiteness of the extension, J should be contain in a minimal prime ideal, that means, $\mathrm{ht} J = 0$. Later we will develop all the necessary facts in order to prove that, if the residue field is algebraically closed, then we can reduce to the case where $a_{ij} \in m$.

REMARK 2.1. Let $R \hookrightarrow S$ be a finite extension of Noetherian rings, where (R, m) is local. Then the maximal spectrum of S, $\mathrm{Spec}_m S$, is equal to $V(mS) \subseteq \mathrm{Spec}\, S$. In fact, since $R/m \hookrightarrow S/mS$ is finite, $\dim S/mS = \dim R/m = 0$. Therefore S/mS is Artinian. Hence $\mathrm{Spec}_m S/mS = \mathrm{Spec}\, S/mS$ is finite. Now, let $\eta \in \mathrm{Spec}_m S$ then $\dim R/(R \cap \eta) = \dim S/\eta = 0$. Hence $R/R\cap\eta$ is a field (a domain of dimension zero), and $R\cap\eta = m$. Consequently $\mathrm{Spec}_m S = V(mS)$.

LEMMA 2.2. *Let (R, m, k) be a local complete ring and $R \hookrightarrow S$ a finite extension. Assume that $\mathrm{Spec}\, S = V(mS) = \{\eta_1, ..., \eta_n\}$. Then S is naturally isomorphic, as a ring, to $S_{\eta_1} \times \cdots \times S_{\eta_n}$.*

PROOF. By Remark 2.1 and the comments made at the beginning of this section, $S/mS = (S/mS)_{\eta_1} \times \ldots \times (S/mS)_{\eta_n}$. But this is equivalent to the existence of idempotent orthogonal elements $e_1, ..., e_n \in S/mS$, which means that, $e_i^2 = e_i$, $\sum_{i=1}^n e_i = 1$ and $e_i e_j = 0$ for all $i \neq j$. In fact $e_i \notin \eta_i$ and $e_i \in \eta_j$ for all $i \neq j$. Let $p(t) = t^2 - t \in S[t]$. Then $\overline{p}(t) = (t - e_i)(t - (1 - e_i)) \in (S/mS)[t]$ for all i, and $(t - e_i, t - (1 - e_i)) = (1)$, because $(t-e_i)-(t-(1-e_i)) = 1-2e_i$ and $e_i(t-e_i)-e_i(t-(1-e_i)) = -e_i^2 = -e_i$, then $1 \in (t - e_i, t - (1 - e_i))$. By Hensel's Lemma (see [**10**, Theorem 7.18.]) there exist linear monic polynomials $t - E_i, t - D_i \in S[t]$ such that $t - \overline{E_i} = t - e_i$ and $t - \overline{D_i} = t - (1 - e_i)$ in $(S/mS)[t]$ (i.e. $\overline{E_i} = e_i$ and $\overline{D_i} = 1 - e_i$) and $p(t) = (t-E_i)(t-D_i)$. Let's fix such $E_i \in R$ for $i = 1, ..., n-1$. Then $p(E_i) = E_i^2 - E_i = 0$, that is, $E_i^2 = E_i$. Besides, for $i \neq j$, $(E_i E_j)^n = E_i^n E_j^n = E_i E_j \in mS$. Thus, for any maximal ideal of S, say η_r, $E_i E_j/1 \in \cap_{m=1}^n (\eta_i S_{\eta_i})^n \subseteq S_{\eta_i}$, because $mS \subseteq \eta_i$ by Remark 2.1. Therefore, by Krull's Intersection Theorem $\cap_{m=1}^n (\eta_i S_{\eta_i})^n = (0)$ (see [**10**, Corollary 5.4.]). Hence, there exists $s \notin \eta_i$ such that $sE_i E_j = 0$, which means that $(0 : E_i E_j) \nsubseteq \eta_r$ and that holds for all maximal ideals η_r. In conclusion, $(0 : E_i E_j) = S$, so $E_i E_j = 0$. Define $E_n = 1 - \sum_{i=1}^{n-1} E_i$. Then

$$E_n^2 = 1 - 2(\sum_{i=1}^{n-1} E_i) + (\sum_{i=1}^{n-1} E_i)^2 = 1 - 2(\sum_{i=1}^{n-1} E_i) + \sum_{i=1}^{n-1} E_i^2 = 1 - \sum_{i=1}^{n-1} E_i = E_n,$$

and

$$E_jE_n = E_i - \sum_{i=1}^{n-1} E_jE_i = E_j - E_j^2 = 0,$$

for all $j < n$, and $\overline{E_n} = 1 - \sum_{i=1}^{n-1} \overline{E_i} = 1 - \sum_{i=1}^{n-1} e_i = e_n$. Therefore $\{E_1, ..., E_n\}$ is a set of idempotent orthogonal elements for S.

Now, $E_i \in \cap_{i\neq j}\eta_i \setminus \eta_j$, because $\overline{E_i} = e_i \in \cap_{i\neq j}\overline{\eta_i} \setminus \overline{\eta_j}$. So, $S_{E_i} = S[e_i^{-1}]$ is a local ring with maximal ideal η_i^e because η_i is the only maximal ideal not containing it. Then $S[e_i] \cong S_{\eta_i}$. Furthermore, there are natural homomorphism of rings $\alpha : S \to SE_i$ sending $E_i \to E_i^2 = E_i$. So α send E_i to the unity on SE_i, and then α induces a homomorphism from S_{E_I} to SE_i which is clearly bijective. Thus, $S_{E_I} \cong SE_i$. In conclusion, there is a natural isomorphism as follows $S \cong \oplus_{i=1}^n SE_i \cong \oplus_{i=1}^n S_{E_i} \cong \oplus_{i=1}^n S_{\eta_i}$.

□

COROLLARY 2.3. *Let (R, m, k) be a local ring with algebraically closed field k and $B = R[x]/(F(x))$, where $F(x)$ is a monic polynomial of degree n. Then there exist monic polynomials $G_i(x)$ of degree n_i such that $\sum_{i=1}^r n_i = n$, $B = \oplus_{i=1}^r R[x]/(G_i(x))$ as rings, and*

$$G_i(x) = x^{n_i} + a_{i1}x^{n_i-1} + \cdots + a_{in_i},$$

where all $a_{ij} \in m$.

PROOF. Since k is algebraically closed we can factor $f(x) := \overline{F}(x) = \prod_{i=1}^r (x - b_i)^{n_i} \in k[x]$, for some $b_i \in k$ and $n_i \in \mathbb{N}$, with $\sum_{i=1}^r n_i = n$. Besides, $B/mB \cong k[x]/(f(t))$ is an Artinian ring with maximal ideal $\overline{\eta_i} = (x-b_i)$ for $i = 1, ..., r$, therefore $B/mB \cong \oplus_{i=1}^r k[x]/((x - b_i)^{n_i})$ (see [**1**, Theorem 8.7 and proof]). Now, by Hensel's Lemma there exist monic polynomials $F_i(x) \in R[x]$ such that $F[x] = \prod_{n=1}^r F_i(x)$ and $\overline{F_i}(x) = (x - b_i)^{n_i}$. By Remark 2.1, $\operatorname{Spec}_m B = \{\eta_1, ..., \eta_r\} = V(mB)$.

Now, let $B_i \in R$ such that $\overline{B_i} = b_i$. Since $\eta_i = (x - b_i)$ in B/mB then we see by the correspondence between the ideals of B/mB an the ideals of B containing mB that $\eta_i = (x - B_i) + mB$. Now, $(F(x))R[x]_{\eta_i} = (F_i(x))R[x]_{\eta_i}$, because $(F_j(x)) + \eta_i = R[x]$ for all $i \neq j$, since mod $mR[x]$ this ideal corresponds to $((x - b_i)^{n_j}) + (x - b_i) = \operatorname{rad}(x - b_i, x - b_j) = k[x]$ for all $i \neq j$. Therefore $F_j(x)$ is a unit in $R[x]_{\eta_i}$. Besides, the ring $R[x]/(F_i(x))$ is local with maximal ideal $(x - B_i) + m^e$, (which we denote again by η_i). This is because any maximal ideal should contain the expansion of m (the extension $R \hookrightarrow R[x]/(F_i(x))$ is finite) and this ring module m^e, is the local ring $(k[x]/((x - b_i)^{n_i}), (x - b_i)^e)$. So,

$$B_{\eta_i} = R[x]_{\eta_i}/(F(x))R[x]_{\eta_i} \cong R[x]_{\eta_i}/F_i(x)R[x]_{\eta_i}$$

$$\cong (R[x]/(F_i(x)))_{\eta_i} \cong R[x]/(F_i(x)).$$

Thus, by the previous lemma, $B \cong \oplus_{i=1}^r B_{\eta_i} \cong \oplus_{i=1}^r R[x]/(F_i(x))$.

Finally, the isomorphism of rings $\theta_i : R[x] \to R[t]$, sending $x \mapsto t + B_i$, induces an isomorphism of rings $\overline{\theta_i} : R[x]/(F_i(x)) \to R[t]/(F_i(t + B_i))$. But the reduction of $F_i(x)$ mod $mR[x]$ is exactly $((t + B_i) - B_i)^{n_i} = t^{n_i}$, because translation and reduction mod m commutes. But that means exactly that $G_i(t) = t^{n_i} + a_{i,1}t^{n_i-1} + \cdots + a_{i,n_i} \in R[t]$ with $a_{i,j} \in m$, for any indeces i, j. In conclusion, we get an isomorphism of rings between B and $\oplus_{i=1}^r R[x]/(G_i(t))$ satisfying the conditions of our corollary. □

REMARK 2.4. Let $i : R \hookrightarrow S = S_1 \times ... \times S_n$ be an extension of rings, where R is and integral domain. Then there exists a S_i such that $\pi_i \circ i : R \hookrightarrow S_i$ is also an extension, where π_i is the natural projection. Suppose by contradiction that for any i there exist $a_i \neq 0 \in R$ such that $\pi_i(i(a_i)) = 0$. Then, if $a = \prod_{i=1}^n a_i \neq 0$, for any j,

$$\pi_j(a_i) = \prod_{i=1}^n \pi_j(i(a_i)) = \pi_j(i(a_j)) \prod_{i \neq j} \pi_j(i(a_i)) = 0.$$

Therefore $i(a) = (\pi_1(i(a)), ..., \pi_n(i(a)) = 0$, contradicting our hypothesis.

THEOREM 2.5. *Let (R, m, k) be a regular complete local ring with algebraically closed residue field k. Then, to prove the DSC for R it is enough to consider finite extensions $R \hookrightarrow S$, where $S = T/J$,*

$$T = R[y_1, ..., y_r]/(f_1(y_1), ..., f_r(y_r)),$$

$f_i(y_i) = y_i^{n_i} + a_{i,1}y_1^{n_i-1} + \cdots + a_{i,n_i}$ with $a_{i,j} \in m$ and $J \subseteq T$ is an ideal of height zero.

PROOF. Let us fix a finite extension $R \hookrightarrow S$. By the discussion above, we know that $S = T/J$, where $T = R[y_1, ..., y_r]/(f_1(y_1), ..., f_r(y_r))$ (the coefficients of the monic polynomials $f_i(y_i)$ are not necessarily in m), and $\mathrm{ht}(J) = 0$. It is elementary to see that

$$T \cong \otimes_{i=1}^r R[y_i]/(f_i(y_i)).$$

Write $B_i = R[y_i]/(f_i(y_i))$ then by Corollary 2.3 $B_i \cong \oplus_{\alpha=1}^{m_i} R[y_i]/(f_{i\alpha}(y_i))$ with $f_{i\alpha}(y_i) = y_i^{n_{i\alpha}} + a_{i\alpha 1}y_i^{n_{i\alpha}-1} + \cdots + a_{i\alpha n_{i\alpha}}$ and $a_{i\alpha j} \in m$. Furthermore, by the distributive law between tensor products and direct sums (see [**30**]) we get

$$T \cong \otimes_{i=1}^r (\oplus_{\alpha=1}^{m_i} R[y_i]/(f_{i\alpha}(y_i))) \cong \oplus_w (\otimes_{i=1}^r R[y_i]/(f_{i\alpha_i}(y_i))) \cong \oplus_w T_w,$$

where $w = (\alpha_1, ..., \alpha_r)$, $1 \leq \alpha_i \leq m_i$ and

$$T_w \cong \otimes_{i=1}^r R[y_i]/(f_{i\alpha_i}(y_i)) \cong R[y_i, ..., y_r]/(f_{1\alpha_1}(y_1), ..., f_{r\alpha_r}(y_r))$$

where each $f_{i\alpha_i}(y_i)$ has lower coefficients in m, as desired. Besides, $J = \oplus_w J_w$ and then $S \cong \oplus_w T_w/J_w$.

Finally, by Remark 2.4, there is an α such that $R \hookrightarrow T_w/J_w$ is an extension. But if there is a retraction $\rho_w : T_w/J_w \hookrightarrow R$, then $\rho = \pi_w \circ \rho_w : S \to R$ is also a retraction. Besides $ht(J_w) = 0$, because $\dim T_w = \dim R = \dim T_w/J_w$. In conclusion, $S_w = T_w/J_w$ has the desired form of our proposition and then it is enough to prove the DSC in this case. □

PROPOSITION 2.6. *Let (R, m, k) be a regular local ring of dimension d, and*

$$T = R[y_1, ..., y_r]/(f_1(y_1), ..., f_r(y_r)),$$

$f_i(y_i) = y_i^{n_i} + a_{i,1}y_1^{n_i-1} + \cdots + a_{i,n_i}$, with $a_{i,j} \in m$. Then T is a Gorenstein local ring with maximal ideal $\eta = m + (\overline{y_1}, ..., \overline{y_r})$.

PROOF. First we see that T is a local C-M ring. In fact, let m_1 be any maximal ideal of T. Then $m_1 \cap R = m$, because $\dim(R/(m_1 \cap R)) = \dim(T/m_1) = 0$ and $m_1 \cap R \in \operatorname{Spec}(R)$. Therefore $R/(m_1 \cap R)$ is a field. Thus, $y_i^{n_i} \in m_1$ and therefore $y_i \in m_1$. In conclusion, $m_1 = \eta$. Now, $R[y_1, ..., y_r]$ is a C-M ring (see [**10**, Proposition 18.9]). Thus $B = R[y_1, ..., y_r]_\eta$ is also C-M. Let $m = (x_1, ..., x_d)$, where $d = \dim R$. Then $\{f_1(y_1), ..., f_r(y_r), x_1, ..., x_d\} \subseteq B$ is a system of parameters because $\dim B = ht(\eta) = \dim R[y_1, ..., y_r] = \dim R + r = d + r$, since $R[y_1, ...y_r]$ is C-M then by previous comments is equidimensional, and $\operatorname{rad}(f_1, ..., f_r, x_1, ..., x_d) = \eta B$. Then $\{f_1, ..., f_r, x_1, ..., x_d\}$ is a regular sequence in B. Thus, $B/(f_1, ..., f_r) \cong T$ is C-M, due to the fact that $\dim B/(f_1, ..., f_r) = d$, and $\{\overline{x_1}, ..., \overline{x_d}\} \subseteq B/(f_1, ..., f_r)$ is a regular sequence.

Finally, let $Q = (y_1, ..., y_r, x_1, ..., x_d)$ be the ideal generated by the system of parameters $\{y_1, ..., y_r, x_1, ..., x_d\}$. Then $T/Q \cong k[y_1, ..., y_r]/(y_1^{n_1}, ..., y_r^{n_r}) = k[w_1, ..., w_r]$, where $w_i = \overline{y_i}$. Let us see that

$$\operatorname{Ann}_{T/Q}((w_1, ..., w_r)) = (\prod_{i=1}^{r} w_i^{n_i-1}).$$

In fact, if $\overline{h} \in \operatorname{Ann}_{T/Q}((w_1, ..., w_r))$ and $c\prod_{i=1}^r w_i^{m_i} \neq 0$ is a monomial of $\overline{h}$ such that there exists $m_j \in \mathbb{N}$ with $m_j < n_j - 1$, then $w_j\overline{h} \neq 0 \in T/Q$, because the monomial term $w_j c \prod_{i=1}^r w_i^{m_i}$ has the power $m_j + 1 < n_j$ on w_j, which is a contradiction, by the reason that h is on the socle. Therefore, $m_i \geq n_i - 1$ for all i and so $\overline{h} \in (\prod_{i=1}^r w_i^{m_i})$. The other contention is clear, and then the socle has dimension one. In conclusion, (T, η) is a Gorenstein local ring. □

3. The DSC in terms of Annihilators

Now we make preparations for the proof of the following fact: let $h : (R, m) \to (T, \eta)$ be a finite homomorphism of local rings, i.e. $h(m) \subseteq \eta$ where T is a local $R-$free ring, with T/mT Gorenstein, and let $S = T/J$, for some ideal $J \subseteq T$. Then $h : R \to S$ splits if and only if $\operatorname{Ann}_T J \nsubseteq mT$ (by abuse of notation we denote by h again its composition with the natural projection $\pi : T \to S$).

PROPOSITION 3.1. *Let (A, η, k) be a local Gorenstein ring of dimension zero (i.e. $\dim_k(\mathrm{Ann}_A \eta) = 1$). Let $u \in \eta$ such that $(u) = \mathrm{Ann}_A \eta$. Then $u \in I$ for any ideal $I \neq (0) \subseteq A$.*

PROOF. Clearly, we can assume that $I \subseteq \eta$. We know $\mathrm{nil}(A) = \eta$, therefore there exists $n \in \mathbb{N}$ such that $\eta^n = 0$. Let $x \neq 0 \in I$. Then $\eta^{n-1}x \subseteq \eta^n = (0)$. Let $r \in \mathbb{N}$ be such that $\eta^r x = (0)$ but $\eta^{r-1}x \neq (0)$. Hence $\eta(\eta^{r-1}x) = (0)$ and so $\eta^{r-1}x \subseteq \mathrm{Ann}_A \eta = (u)$. Then (u) contains a nonzero element of the form bx, where $b \in \eta^{r-1}$. That means that there exists $c \in A \smallsetminus \eta$ with $cu = xb$, so $u = (c^{-1}b)x \in I$. □

REMARK 3.2. If $h : R \hookrightarrow T$ is any homomorphism of rings, we can consider $T^* = \mathrm{Hom}_R(T, R)$ as a $T-$module with the following action: fix $t \in T$ and define $(t \cdot \phi)(x) := \phi(tx)$, for $\phi \in \mathrm{Hom}_R(T, R)$ and $x \in T$.

REMARK 3.3. Let (R, m) be a local ring, T a finitely generated $R-$free module, and $\theta : T \to T$ an $R-$homomorphism. Then θ is an isomorphism of $R-$modules if and only if $\overline{\theta} : T/mT \to T/mT$ is an isomorphism of $K-$vector spaces. In fact, if $A \in M_{n\times n}(R)$ is the matrix defining θ, then θ is an isomorphism if and only if $\det A$ is an unit, which means that $\det A \notin m$. But that is equivalent to saying that $det\overline{A} \neq 0 \in k$, where $\overline{A}$ is the reduction of A mod m. Finally, since $\overline{A}$ is the matrix defining $\overline{\theta}$, the last condition is equivalent to saying that $\overline{\theta}$ is an isomorphism of $k-$vector spaces.

THEOREM 3.4. *Let (R, m) and (T, η) be local rings. Assume that T/mT is Gorenstein. Let $h : R \to T$ be a finite homomorphism of local rings, such that T is $R-$free. Then there exists a $T-$isomorphism $\beta : T \to T^*$ such that for any ideal $J \subseteq T$, $\beta^{-1}((T/J)^*) = \mathrm{Ann}_J T$.*

PROOF. We identify $(T/J)^*$ with $\{f \in T^* : f(J) = 0\}$. We know that $\dim T/mT = \dim R/m = 0$, since T/mT is a finitely generated $R/m-$module. So, fix $u_1 \in \eta$ such that $(\overline{u_1}) = \mathrm{Ann}_{T/m}\overline{\eta}$, since T/mT is Gorenstein. Let $\{u_2, ..., u_d\} \subseteq T$ such that $T/mT = (\overline{u_1}, ..., \overline{u_d})$ (where $T \cong R^d$). Then, by the Lemma of Nakayama $T = (u_1, ..., u_d)$, T is generated by $u_1, ..., u_d$ as an $R-$free module. In fact, let $\{w_1, ..., w_d\} \subseteq T$ be an $R-$basis for T. Define $\theta : T \to T$ by $w_i \to u_i$, then the induced $\overline{\theta} : T/mT \to T/mT$ is clearly an isomorphism of $k-$vector spaces. Since T is $R-$free, by Remark 3.3, θ is an isomorphism which means just that $\{u_1, ..., u_d\} \subseteq T$ is an $R-$basis for T. Let $u_1^* \in T^*$ be the dual element and define $\beta : T \to T^*$ by $t \to t \cdot u_1^*$, where $(t \cdot u_1^*)(t_1) := u_1^*(tt_1)$, for all $t_1 \in T$. By definition, it is clear that β is an $T-$homomorphism. Now, we can make the natural identifications $T^* = \mathrm{Hom}_R(R^d, R) \cong (\mathrm{Hom}_r(R, R))^d = R^d$. Therefore, T^* is an $R-$free module of dimension d and β is an $R-$isomorphism if and only if $\overline{\beta} : T/mT \to T^*/mT^*$ is so, due to Remark 3.3 ($T \cong T^*$ as $R-$free modules). But $\overline{\beta}$ is an isomorphism of $k-$vector spaces if it is injective. Suppose by contradiction that $\ker\overline{\beta} \neq (0)$.

Then, by Proposition 3.1, $\overline{u_1} \in \ker\overline{\beta}$. That means, $\beta(u_1) \in mT^*$, so there exist $m_1, ..., m_d \in m$ such that $\beta(u_1) = u_1u_1^* = \sum_{i=1}^d m_iu_i^*$. But, evaluating at 1 we get: $1 = u_1u_1^*(1) = \sum_{i=1}^d m_iu_i^*(1) \in m$, a contradiction. In conclusion, β is a $T-$isomorphism.

For the last part, let $a \in \mathrm{Ann}_T J$. Then, for any $j \in J$, $\beta(a)(j) := (au_1^*)(j) = u_1^*(aj) = u_1^*(0) = 0$. Therefore, $\beta(a) \in (T/J)^*$ and thus $a = \beta^{-1}(\beta(a)) \in \beta^{-1}((T/J)^*)$.

Conversely, take $\phi_0 \in \beta^{-1}((T/J)^*)$. Then there exists a $\phi \in (T/J)^*$ such that $\phi_0 = \beta^{-1}(\phi)$, i.e. $\phi = \phi_0u_1^*$ and $\phi(J) = 0$, which means that $u_1^*(\phi_0 j) = 0$ for all $j \in J$. Now, let's fix $j_0 \in J$. Then, for all $t \in T$, $\beta(\phi_0j_0)(t) = u_1^*(\phi_0j_0t) = u_1^*(\phi_0(j_0t)) = 0$, because $j_0t \in J$. Therefore $\beta(\phi_0j_0) \equiv 0$ and then, by the first part $\phi_0j_0 = 0$, which means that $\phi_0 \in \mathrm{Ann}_T J$. □

THEOREM 3.5. *Let $h : R \to T/J$ be a finite homomorphism of local rings such that (T, η) is a local $R-$free ring, with T/mT Gorenstein, and $J \subseteq T$ an ideal. Assume that the structure of $R-$module of T/J inherited by the $R-$structure of T is the same as the one induced by h. Then $R \hookrightarrow T/J$ splits if and only if* $\mathrm{Ann}_T J \nsubseteq mT$.

PROOF. Assume that $\rho : T/J \hookrightarrow R$ is a splitting $R-$homomorphism. Then $\rho \in (T/J)^*$ and $\rho(1) = 1$. By the last theorem, there exists $\rho_0 \in \mathrm{Ann}_T J$ such that $\rho_0 = \beta^{-1}(\rho)$ which means, in particular, that $1 = \rho(1) = \rho_0 \cdot u_1^*(1) = u_i^*(\rho_0) \notin m$. Then $\rho_0 \notin mT$, so $\mathrm{Ann}_T J \nsubseteq mT$.

Conversely, take $a \in \mathrm{Ann}_T J \setminus mT$. Then, by Proposition 3.1, $\overline{u_1} \in (\overline{a}) \subseteq T/mT$, which means that there exists $m_i \in m$, with $u_1 - a = \sum_{i=1}^d m_iu_i$, where $\{u_1, ..., u_d\} \subseteq T$ is an $R-$basis for T as in the last Theorem. Then,

$$au_i^*(1) = (1 - m_1)u_1u_1^*(1) - \sum_{i=2}^d m_iu_iu_i^*(1)$$

$$= (1 - m_1) - \sum_{i=2}^d m_iu_1^*(u_i) = (1 - m_1) + 0 = 1 - m_1 \notin m.$$

So $\rho = \beta((1 - m_i)^{-1}a)$ satisfies that $\rho \in (T/J)^*$ because $\beta^{-1}(\rho) = (1 - m_1)^{-1}a \in \mathrm{Ann}_T J$, and by Theorem 3.4 $\beta^{-1}((T/J)^*) = \mathrm{Ann}_T J$. Besides, $\rho_1 = (1 - m_i)^{-1}au_1^*(1) = (1 - m_1)^{-1}(1 - m_1) = 1$, which implies that $\rho : T/J \to R$ is the desired splitting $R-$homomorphism. □

4. Reduction to the case where J is principal

In the next proposition we will prove that we can reduce to the case where J is a principal ideal generated by an element in mT.

Proposition 4.1. *Let* (R_0, m_0, k_0) *be a regular local ring and* $R_0 \hookrightarrow T_0$ *be a finite extension, where*

$$T_0 = R_0[y_1, ..., y_r]/(f_1(y_1), ..., f_r(y_r)),$$

and each $f_i(y_i) = y_i^{n_i} + a_{i1}y_i^{n_i-1} + \cdots + a_{in_i}$, *with* $a_{ij} \in m$, *for all indices i,j. Let* $S = T_0/J$, *with* $J = (g_1, ..., g_s) \subseteq T_0$ *such that* $J \cap R_0 = (0)$. *Let* $x_1, ..., x_s$ *be new variables, and let* R *be* $R_0[x_1, ..., x_s]$,*and let* m *be* $m_0 + (x_1, ..., x_s)$, *a maximal ideal of* R. *Let* g *be the element* $x_1g_1 + \cdots + x_sg_s$. *Write*

$$T = R \otimes_{R_0} T_0 \cong R[y_1, ..., y_r]/(f_1(y_1), ..., f_r(y_r)).$$

Then:

(1) (R_m, mR_m, k_0) *is a regular local ring.*
(2) $(g)T_m \cap R_m = (0)$.
(3) $R_m \hookrightarrow (T/(g))_m \cong T_m/(g)^e$ *is a finite extension, and* $g \in (mR_m)T_m$.
(4) $R_m \hookrightarrow T_m/(g)$ *splits if and only if* $R_0 \hookrightarrow T_0/J$ *splits.*

Proof. (1) In general, if R is regular then so is the polynomial ring $R[T]$ (see [**30**]). In particular, R_m is a regular local ring and $R_m/mR_m \cong R/m \cong R_0/m_0 = k_0$.

(2) $R_0 \hookrightarrow R$ is an $R-$free extension, then, in particular, it is flat. Therefore, by tensoring $R_0 \hookrightarrow T_0/J$ with R, we see that $R \hookrightarrow R \otimes_{R_0} T_0/J \cong T/J^e$ is also an extension, and since localization is flat too, we get an extension $R_m \hookrightarrow T_m/J^e$. Because of $g \in J^e$, we get $gT_m \cap R_m = 0$.

(3) Clearly, by definition $g \in (mR_m)T_m$. Now, by the previous paragraph $R_m \hookrightarrow T_m/(g)^e$ is an extension, and it is finite because $R_0 \hookrightarrow T_0$ is finite, in fact free. Then, after tensoring with R_m we get a module finite extension $R_m \hookrightarrow T_m$, so $T_m/(g)^e$ is also a finitely generated R_m-module.

(4) Assume that $\rho_0 : T_0/J \to R_0$ is an R_0-homomorphism such that $\rho_0(1) = 1$. Then, by tensoring with the flat R_0-module R_m, we get an R_m-homomorphism $\rho : T_m/J \hookrightarrow R_m$, with $\rho(1) = 1$. Now, composing ρ with the natural map $T_m/(g) \to T_m/J$, we obtain a retraction from $T_m/(g)$ to R_m.

Conversely, it is clear that T_m satisfies the hypothesis of Proposition 2.6. Therefore, by Theorem 3.5, $\mathrm{Ann}_{T_m}(g) \nsubseteq (mT_m$. So let's choose

$$w = \sum_\alpha (h_\alpha(\underline{x})/k_\alpha(\underline{x}))y^\alpha,$$

such that $wg = 0$, where $h_\alpha \in R$ and $k_\alpha \in R \setminus m$ (which is equivalent to saying that $k_\alpha(\underline{0}) \notin m_0$). Here y^α denotes $y_1^{\alpha_1}...y_r^{\alpha_r}$, $\alpha = (\alpha_1, ..., \alpha_r)$, $0 \leq \alpha_i < degf_i$ and some $h_\beta \notin m$ (that means exactly $w \notin mT_m$). We have $T = R \otimes_{R_0} T \cong T_0[x_1, ..., x_s]$. Now, multiplying by the product of the $k_\alpha(\underline{x})$, we can assume that $w = \sum_\alpha p_\alpha(\underline{x})y^\alpha \in T$, where some $p_\alpha \notin m$ and $0 = wg = \sum_i \sum_\alpha p_\alpha(\underline{x})y^\alpha x_i g_i(\underline{y})$ in T. Now, the coefficient of x_i, which is zero in T, is exactly $\sum_\alpha p_\alpha(\underline{0})y^\alpha g_i(\underline{y})$, because the terms $y^\alpha g_i(\underline{y})$ are constants in $T = T_0[x_1, ..., x_s]$. Therefore, if $w_0 = \sum_\alpha p_\alpha(\underline{0})y^\alpha \in T_0$, we have $w_0g_i =$

$\sum_\alpha p_\alpha(\underline{0})y^\alpha g_i(\underline{y}) = 0$, and thus $w_0 \in \mathrm{Ann}_{t_0} J$. But $p_\beta(\underline{0}) \notin m_0$, because $p_\beta(\underline{x}) \notin m$, so $w_0 \notin m_0T_0$. In conclusion, $\mathrm{Ann}_{T_0} J \nsubseteq m_0T_0$, which is equivalent by Theorem 3.5 to the fact that $R_0 \hookrightarrow T_0/J$ splits.

□

5. The Socle-Parameters Conjecture

In this and the next section we state two new conjectures (The Socle-Parameters Conjecture: Strong Form (SPCS) and Weak Form (SPCW)) and we will prove that the SPCW is equivalent to the DSC and that SPCS implies SPCW. Besides, these two conjectures are equivalent in the equicharacteristic case and therefore both are equivalent to the DSC in the equicharacteristic case, which is a theorem (see previous comments). However, the mixed characteristic case remains open. The new approach shows that the DSC is in essence a problem concerning algebraic and homological properties of Gorenstein local rings.

Socle-Parameters Conjecture, Strong Form (SPCS). Let (T, η) be a Gorenstein local ring of dimension d. Let $\{x_1, ..., x_d\} \subseteq T$ be a system of parameters and write $Q = (x_1, ..., x_d)$. Let $u \in T$ be any lifting of a socle element in T/Q, i.e. $\mathrm{Ann}_{T/Q}(\bar{\eta}) = (\bar{u})$. Let $z \in T$ be a zero divisor. Then $u \cdot z \in Q \cdot (z)$. This is equivalent to saying that $\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0$.

Now, we prove the last equivalence:

Proposition 5.1. *In the situation of the SPCS the following are equivalent.*

(1) $u \cdot z \in Q \cdot (z)$.
(2) $\mathrm{Ann}_T(z) \nsubseteq Q$.
(3) $\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0$.

Proof. (1) ⇒ (2) Consider the following natural short exact sequence

$$0 \longrightarrow \mathrm{Ann}_T(z) \longrightarrow T \longrightarrow (z) \longrightarrow 0.$$

We know that $T/\mathrm{Ann}_T(z) \cong (z)$, by the isomorphism sending $\bar{t}$ to tz. After tensoring with T/Q we get

$$(z)/(Q(z)) \cong T/Q \otimes (z) \cong T/Q \otimes T/\mathrm{Ann}_T(z) \cong T/(\mathrm{Ann}_T(z) + Q)).$$

Now, $uz \in Q\cdot(z)$ if and only if $\mathrm{Ann}_T(z) \nsubseteq Q$. Effectively, $uz \in Q\cdot(z)$ is equivalent to $\overline{uz} = 0 \in (z)/(Q(z))$, and it is equivalent to $\overline{u} = 0 \in T/(\mathrm{Ann}_T(z)+Q)$, under the last isomorphism. Therefore, there exists $w \in \mathrm{Ann}_T(z)$ and $q \in Q$ such that $u = w+q$, and so $w = u-q \notin Q$ because $u \notin Q$. Then $\mathrm{Ann}_T(z) \nsubseteq Q$.

(2) ⇒ (1) If $\mathrm{Ann}_T(z) \nsubseteq Q$, by Proposition 3.1, $\overline{u} \in \overline{\mathrm{Ann}_T(z)} \subseteq T/Q$. Then there exists $w \in \mathrm{Ann}_T(z)$ such that $\overline{u} = \overline{w}$, which means that there is a $q \in Q$ such that $u = w + q$. So,

$$uz = (w + q)z = wz + qz = qz \in Q \cdot (z).$$

(2) $\Leftrightarrow$ (3) $\mathrm{Ann}_T(z) \nsubseteq Q$ if and only if $Q \subsetneq \mathrm{Ann}_T(z) + Q$ if and only if $\ell(T/(\mathrm{Ann}_T(z) + Q)) < \ell(T/Q)$. Besides, we have the natural short exact sequence

$$0 \longrightarrow (z) \longrightarrow T \longrightarrow T/(z) \longrightarrow 0.$$

Then, after considering the induced long exact sequence for Tor, and noting that $\mathrm{Tor}_1^T(T, T/Q) = 0$, because T is a $T-$free module, and therefore flat (see previous results), we get the following exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^T(T/(z), T/Q) \longrightarrow (z)/Q(z) \longrightarrow T/Q \longrightarrow T/(Q + (z)) \longrightarrow 0.$$

Now, since Q is generated by a system of parameters, the $T-$modules $T/(Q+(z))$, T/Q and $T/(\mathrm{Ann}_T(z)) \cong (z)/Q(z)$ are Noetherian rings of dimension zero and therefore Artinian. In particular, they have finite length as $T-$modules. Then the submodule $\mathrm{Tor}_1^T(T/(z), T/Q))$ has finite length too. By the additivity of $\ell(-)$, we have

$$\ell(\mathrm{Tor}_1^T(T/(z), T/Q)) - \ell(T/(\mathrm{Ann}_T(z) + Q)) + \ell(T/Q) - \ell(T/(Q + (z)) = 0.$$

Hence,

$$\ell(T/Q) - \ell(T/(\mathrm{Ann}_T(z) + Q)) = \ell(T/(Q + (z)) - \ell(\mathrm{Tor}_1^T(T/(z), T/Q)).$$

Then, $\mathrm{Ann}_T(z) \nsubseteq Q$ if and only if $\ell(T/(Q + (z)) - \ell(\mathrm{Tor}_1^T(T/(z), T/Q)) > 0$. But T is C-M and then $\{x_1, ..., x_n\}$ is a regular sequence. Hence, the Koszul Complex is a free (and then projective) resolution of T/Q:

$$\cdots \longrightarrow K_2 \longrightarrow K_1 \longrightarrow K_0 \longrightarrow T/Q \longrightarrow 0.$$

Hence, after tensoring this resolution with $T/(z)$, and taking homology, we find that $H_1(\underline{x}, T/(z)) \cong \mathrm{Tor}_1^T(T/Q, T/(z))$ and $H_0(\underline{x}, T/(z)) \cong (T/(z))/\overline{Q}) \cong T/((z) + Q)$. In conclusion, $\mathrm{Ann}_T(z) \nsubseteq Q$ is equivalent to the condition

$$\ell(H_0(\underline{x}, T/(z)) - \ell(H_1(\underline{x}, T/(z)) > 0.$$

$\square$

PROPOSITION 5.2. *Assume the same hypothesis as in SPCS and, in addition, that* $\mathrm{depth}(\overline{\eta}, T/(z)) \geq d - 1$*, then SPCS holds.*

PROOF. Write $a = \mathrm{depth}(\overline{\eta}, T/(z))$. By previous comments (Ch. 1 §1), we know that if $q = \sup\{r : H_r(\underline{x}, T/(z)) \neq 0\}$ then $a = d - q$, therefore $q = d - a \leq 1$. In the case that $d = 0$, $Q = 0$ and $\dim_T(\eta) = 1$, thus for any element $u \in T$ holds $uz = 0 \in Q \cdot (z)$. Then assume $d \geq 1$. Besides, $\{x_1, ..., x_n\} \subseteq T$ is a system of parameters for the $T-$module $T/(z)$, because $\dim T = \dim T/(z)$ and $\dim((T/(z))/(x_1, ..., x_d)T/(z)) = 0$. So, $(T/(z))/(x_1, ..., x_d)T/(z)$ is an Artinian ring. Hence, by previous results, we get

$$\ell(H_0(\underline{x}, T/(z)) - \ell(H_1(\underline{x}, T/(z)) = \chi(\underline{x}, T/(z)) = e(\overline{Q}, T/(z)).$$

Now, by previous comments and the fact that $d \geq 1$ we see that $e(\overline{Q}, T/(z)) > 0$.

□

Socle-Parameters Conjecture, weak Form (SPCW). Let (T, η) be a Gorenstein local ring of dimension d. Let $\{x_1, ..., x_d\}$ be a system of parameters (if T is mixed characteristic ($\mathrm{char} T/\eta = p > 0$), we assume that $x_1 = p$). Let $Q = (x_1, ..., x_d)$, and let $u \in T$ be any lifting of a socle element in T/Q, i.e. $\mathrm{Ann}_{T/Q}(\overline{\eta}) = (\bar{u})$. Let z be a zero divisor. Then $u \cdot z \in Q \cdot (z)$, which is equivalent to $\ell(H_0(\underline{x}, T/(z))) - \ell(H_1(\underline{x}, T/(z))) > 0$.

Note that between the two forms of the SPC the only difference is the fact that in the mixed characteristic case we assume that $x_1 = p$. This condition has a technical nature and it is necessary in order to apply Cohen's structure theorem in mixed characteristic.

Remark 5.3. For proving any of the two versions of the SPC it is enough to assume that (T, η) is complete.

Proof. Let $\tau : T \to \widehat{T}$ be the natural homomorphism to the completion. Then τ is an faithfully flat extension and $I\widehat{T} \cap T = I$ for any ideal I of T (see [**30**, p. 63]). Besides, another elementary consequence of faithfully-flatness is that for any ideals I, J of T, $(J : I)\widehat{T} = (J\widehat{T} : I\widehat{T})$. Now, assume by contradiction that there exists a system of parameters $\{x_1, ..., x_d\}$ (for the SPCW assume $x_1 = p$), a zero divisor $z \in T$ and $u \in T$ a lifting of a socle element for T/Q such that $uz \notin Q(z)$. Let us write $\tau(y) = y'$. Then $\{x'_1, ..., x'_d\} \subseteq \widehat{T}$ is a system of parameters in $\widehat{T}$, because $\hat{\eta} = \eta\widehat{T} = \mathrm{rad}((x_1, ..., x_d)\widehat{T}) \subseteq \mathrm{rad}(x'_1, ..., x'_d) \subseteq \hat{\eta}$. Besides, $(Q \cdot (z) : z)\widehat{T} = (Q\widehat{T} \cdot (z') : z')$; $\widehat{T}/Q\widehat{T} \cong \widehat{(T/Q)} = T/Q$, due to the fact that $\overline{\eta}^n = (0)$ for some $n > 0$; therefore $(u') = \mathrm{Ann}_{\widehat{T}/Q\widehat{T}}(\hat{\eta})$ and so u' is a socle element. Note that $p = \mathrm{char}(\widehat{T}/Q\widehat{T}) = \mathrm{char}(T/Q)$. Furthermore, $\widehat{T}$ is also a Gorenstein ring (see [**30**, Theorem 18.3]).

Finally, $((Q\widehat{T} : z') + (u'))/(Q\widehat{T} : z) \cong ((Q : z) + u))/(Q : z)) \otimes \widehat{T} \neq 0$, because $(Q : z) + u)/(Q : z) \neq 0$, since $uz \notin Q(z)$. Then $u'z' \notin Q\widehat{T} \cdot (z')$ which contradicts SPC in the complete case. □

6. Equivalence to the DSC

Theorem 6.1. *The Socle-parameters Conjecture (weak Form) is equivalent to the Direct Summand Conjecture.*

Proof. *SPCW* $\Rightarrow$ *DSC*. By previous comments we may assume that $R \hookrightarrow S$ is a finite extension and R is a unramified regular local mixed characteristic ring ($\mathrm{char}(R/m) = p > 0$) with algebraically closed residue field

k. Besides, by Theorem 2.5 we can assume that $S = T/(J)$ such that $T = R[y_1, ..., y_r]/(f_1(y_1), ..., f_r(y_r))$; $f_i(y_i) = y_i^{n_i} + a_{i1}y_i^{n_i-1} + \cdots + a_{in_i}$, where $a_{ik} \in m$, for all indexes i,k, and $\mathrm{ht}(J) = 0$. Now, T is a Gorenstein local ring by Proposition 2.6. Moreover, by Proposition 4.1 $R \hookrightarrow S$ splits if and only if $R_1 := R[x_1, ..., x_d]_{m_1} \hookrightarrow S_1 = T_1/(z)$ splits, where $m_1 = m + (x_1, ..., x_d)$, $T_1 = R_1[y_1, ..., y_r]/(f_1(y_1), ..., f_r(y_r))$ and $z \in T_1$.

Besides, by previous results $R[x_1, ..., x_d]$ is a regular ring. In particular R_1 is regular. Furthermore, R_1 is unramified, otherwise there exist elements $a_i, b_i \in m_1$ and $s \in R_1 \setminus m_1$ such that $sp = \sum_{i=1}^{c} a_i b_i$, and then evaluating in $(0, ..., 0)$ we get $p = s(\underline{0})^{-1} \sum_{i=1}^{c} a_i(\underline{0}) B_i(\underline{0})$ where, $a_i(\underline{0}), b_i(\underline{0}) \in m$ and $s(\underline{0}) \in R \setminus m$, so $p \in m^2$, which is a contradiction.

Hence, $p \notin m^2$ and then $\overline{p} \neq 0 \in m/m^2$ is a part of a basis of m/m^2 as k-vector space, which is equivalent by the Lemma of Nakayama to the fact that p is a part of a minimal set of generators of m, say, $\{w_1 = p, ..., w_n\} \subseteq m$. Now, $\{w_1 = p, ..., w_n\} \subseteq T_1$ is a system of parameters in T_1, because $\dim(T_1/m_1T_1) = \dim(r_1/m_1) = 0$ and $\dim T_1 = \dim R_1$, since $R_1 \hookrightarrow S_1$ is a finite extension.

On the other hand, z is a zero divisor en T_1 because $\dim T_1/(z) = \dim S_1 = \dim R_1 = \dim T_1$ and therefore z is contained in a minimal prime of T_1, since T is C-M.

Since T_1 is Gorenstein, choose $u \in T_1$ such that $(\overline{u}) = \mathrm{Ann}_{T_1/m_1T_1}(\overline{\eta})$. By SPCW, $uz \in m_1T_1 \cdot (z)$, so there exists $a \in m_1T_1$ such that $uz = az$, hence $(u - a)z = 0$. But $u - a \notin m_1T_1$, because $u \notin m_1T_1$. Therefore $\mathrm{Ann}_{T_1}(z) \nsubseteq m_1T_1$, and then, by Theorem 3.5, $R_1 \hookrightarrow S_1$ splits.

$DSC \Rightarrow SPCW$. Let (T, η) be a Gorenstein local ring and $\{x_1, ..., x_d\} \subseteq T$ a system of parameters ($x_1 = p$ in the mixed characteristic case). By Remark 5.3 we can assume that T is local. Let D be a coefficient ring for T (which always exits for any complete local ring, see previous results). Then, due to the Cohen's Structure Theorem (see [**8**, Lemma 16]), the ring generated as D-algebra by the parameters $R = D[x_1, ..., x_d]$ is a complete regular local ring with maximal ideal $Q = (x_1, ..., x_d)$ such that the extension $R \hookrightarrow T$ is finite. Since R is regular, then, by Serre's theorem (see [**30**] Theorem 19.2) $pd_R(T)$ is finite. Hence, for the Auslander-Buchsbaum formula and the fact that $\mathrm{depth}(Q, T) = \mathrm{depth}(\eta, T)$ (see [**7**, Exercise 1.2.26]), we know that

$$\mathrm{pd}_R(T) = \mathrm{depth}(Q, R) - \mathrm{depth}(Q, T) =$$

$$\dim R - \mathrm{depth}(\eta, T) = d - \dim T = d - d = 0.$$

So T is a free R-module. Furthermore, z is contained in an associated prime of T because it is a zero divisor. Since T is C-M, then, by previous comments, any associated prime is, in fact, a minimal prime. Thus, z is contained in a minimal prime $P \in \mathrm{Spec}\, T$. Moreover, since T is C-M, T is equidimensional, that means, in particular, that $\dim T/(z) \geq \dim T/P = \dim T$. In conclusion,

$$\dim R/((z) \cap R) = \dim T/(z) = \dim T = \dim R,$$

so $(z) \cap R = (0)$, because R is a domain (R is regular!). Now, to see that $uz \in Q \cdot (z)$, it is enough to see that $\mathrm{Ann}_T(z) \nsubseteq Q$. In fact, if $\mathrm{Ann}_T(z) \nsubseteq Q$ then $\overline{\mathrm{Ann}_T(z)} \neq 0$ in T/Q. Therefore, by Proposition 3.1, $\overline{u} \in \overline{\mathrm{Ann}_T(z)}$, there exists $w \in \mathrm{Ann}_T(z)$ such that $u - w \in Q$. Thus, $uz = (u-w)z \in Q \cdot (z)$, because $wz = 0$. By Theorem 3.5, $R \hookrightarrow T/(z)$ splits if and only if $\mathrm{Ann}_T(z) \nsubseteq mT = Q$, where $m = (x_1, ..., x_d) \subseteq R$. Hence, the DSC for $R \hookrightarrow T/(z)$ implies $\mathrm{Ann}_T(z) \nsubseteq QT$, and then $uz \in QT \cdot (z)$ □

7. The SPCS on Low Multiplicities

The way to attack the SPCS would be by induction on the multiplicity of T, $e(T)$. First, by Remark 5.3 we can assume that T is complete. If $e(T) = 1$ then since T is a complete C-M ring, it is equicharacteristic and therefore unmixed. Hence by the Criterion of multiplicity one (see [**29**]) T is a regular local ring, in particular, an integral domain, which implies $z = 0$, holding directly the SPCS. Suppose that $e(T) = 2$. Since T is C-M, it satisfied the condition $S2$ of Serre, i.e. for any $P \in \mathrm{Spec}T$, $\mathrm{depth}(P, T) \geq \min(2, \dim T_P)$, due to the fact that $\mathrm{depth}(P, T) = \dim(T_P)$. Hence, by a Theorem of Ikeda (see [**26**, Corollary 1.3.]) T is an hypersurface of the form $B/(f)$, where B is a complete regular local ring. Now, we will prove a more general result, namely, that the SPCS holds for residue class ring of local Gorenstein rings which are UFD and C-M, which implies, in particular, the case of multiplicity two because regular local rings are UFD and C-M (see previous comments).

Proposition 7.1. *The SPCS holds for Gorenstein rings of the form $T = B/(f)$, where B is a local C-M ring which is a UFD and $f \neq 0 \in B$.*

Proof. Let $z \neq 0 \in T$ be a zero divisor and $\{\overline{y_1}, ..., \overline{y_d}\} \subseteq T$ a system of parameters. we will see that $\ell(H_0(\underline{y}, T/(z))) - \ell(H_1(\underline{y}, T/(z))) > 0$. The minimal prime ideals of T are just the principal ideals generated by the prime factors of $f = \prod f_i^{c_i}$, i.e. $P_i = (f_i)$, since B is a UFD. Besides, it is enough to prove SPCS for $z = \overline{f_i}$, because each zero divisor is a multiple of one of these, i.e. $z = a\overline{f_i}$ for some $a \in B$, and thus if $u\overline{f_i} \in Q \cdot (\overline{f_i})$, where $Q = (\overline{y_1}, ..., \overline{y_d})$ and $(\overline{u}) = \mathrm{Ann}_{T/Q}(\overline{\eta})$ then $uz = ua\overline{f_i} = ua\overline{f_i} \in Q \cdot (a\overline{f_i}) = Q \cdot (z)$. Let us fix some f_j, then $T/(\overline{f_j}) = B/(f_j)$ is a C-M ring, because is a quotient of a C-M ring by a ideal generated by a regular element (B is an integral domain) (see [**10**, Proposition 18.13]). Since T is equidimensional, $\dim T = \dim T/(f_j)$ and

$$\dim(T/(f_j))/(\overline{y_1}, ..., \overline{y_d}) = \dim B/(f_j, y_1, ..., y_d) \leq$$

$$\dim B/(f, y_1, ..., y_d) = \dim T/(\overline{y_1}, ..., \overline{y_d}) = 0,$$

because $(f_j, y_1, ..., y_d) \subseteq (f, y_1, ..., y_d) \subseteq B$. Hence, $(\overline{y}) = \{\overline{y_1}, ..., \overline{y_d}\} \subseteq T/(f_j)$ is a system of parameters and so it is a regular sequence. Thus $H_1(\underline{y}, T/(\overline{f_j})) = 0$ (see [**30**, Theorem 16.5]). In conclusion,

$$\ell(H_0((\overline{y}), T/(\overline{f_j}))) - \ell(H_1((\overline{y}), T/(\overline{f_j}))) = \ell(H_0((\overline{y}), T/(\overline{f_j})))$$
$$= \ell((T/(\overline{f_j})/(\overline{y_1}, ..., \overline{y_d}))) \geq 1 > 0,$$

because $T/(\overline{f_j})/(\overline{y_1}, ..., \overline{y_d}) \neq (0)$. This prove the SPCS for T.

□

Typical examples of local C-M rings which are UFD are localizations of polynomial rings in prime ideals, or rings of formal power series over DVD or fields. More generally regular local rings fulfill these conditions (see [**10**, Corollary 18.7, Theorem 19.19]).

8. A New Proof of DSC in the Positive Characteristic Case

Now, we present a new proof of the DSC in the positive characteristic case by proving some particular case of SPCW. The new key ingredient is the following Lemma and we refer to [**34**, Proposition 5.2.6] for a proof.

LEMMA 8.1. *Let R be a Noetherian ring, M an $R-$module and let $x_1, ..., x_n$ be a sequence of elements in R such that $M/(x_1, ..., x_n)M$ has finite length. Let $i \in \{1, ..., n\}$. Then, there exists a constant c such that the length of the Koszul homology module $\ell(H_i(x_1^q, ..., x_n^q; M)) \leq cq^{n-i}$, for all $q \in \mathbb{N}_{>0}$.*

PROPOSITION 8.2. *Let (R, m, k) be an equicharacteristic regular local ring, with $\mathrm{char}R = p > 0$, and $R \hookrightarrow S$ a finite extension. Then $R \hookrightarrow S$ splits.*

PROOF. After tensoring with the completion of R, which is faithfully flat, we can assume, by previous comments, that R is complete. By Cohen's Structure Theorem (see [**19**, Theorem, p. 26]), $R \cong k[[x_1, ..., x_n]]$, where $\mathrm{char}k = p > 0$. Now, we can assume that k is perfect (i.e. $k^p = k$), because each extension of the tower $R \hookrightarrow \overline{k} \otimes_k R \hookrightarrow \overline{k}[[x_1, ..., x_n]]$ is faithfully flat, where $\overline{k}$ denotes an algebraic closure of k. Effectively, $R \hookrightarrow \overline{k} \otimes_k R$ is $R-$free and therefore faithfully flat. Besides, we can identify $\overline{k} \otimes_k R$ with $\cup_{i \in I} E_i[[x_1, ..., x_n]]$, where E_i runs over all field extensions $k \subseteq E_i \subseteq \overline{k}$, such that $[E : k] < +\infty$. From this, we see that the completion of the local ring $\overline{k} \otimes_k R$ is exactly $\overline{k}[[x_1, ..., x_n]]$ and so $\overline{k} \otimes_k R \hookrightarrow \overline{k}[[x_1, ..., x_n]]$ is faithfully flat.

Again, by Theorem 2.5 and Proposition 2.6 we can assume that $S \cong T/J$, where T is a Gorenstein local ring and ht $J = 0$. Moreover, by Proposition 4.1 and after tensoring with the completion of $R_1 = R[w_1, ..., w_r]_{(m+(w_1,...,w_r))}$, which is isomorphic to $\overline{k}[[X_1, ..., x_m]]$ (for some $m \geq n$), we see that $\widehat{R_1} \otimes_{R_1} (R_1 \otimes_R T)$ has exactly the same form as in Proposition 2.6. But, now we can assume that J is a principal ideal generated by a zero divisor. In conclusion, we can assume that $R = k[[x_1, ..., x_m]]$, where k is a perfect field and $S = T/(z)$, where T is a Gorenstein local ring and z is a zero divisor.

Now, we set $P_q = (x_1^q, ..., x_n^q) \subseteq T$, where q is a power of p. Note that $R^q = k[[x_1^q, ..., x_n^q]] \hookrightarrow R$ is finite and thus by the proof of Theorem 6.1 $R^q \hookrightarrow S$ splits if and only if

$$\delta = \delta_{R^q}(x_1^q, ..., x_n^q; T/(z)) :=$$
$$\ell_{R^q}(H_0(x_1^q, ..., x_n^q; T/(z)) - \ell_{R^q}(H_1(x_1^q, ..., x_n^q; T/(z)) > 0.$$

Besides, it is elementary to see that

$$\delta = \delta_R(x_1^q, ..., x_n^q; T/(z)),$$

because the degree of the residue field extension of $R^q \hookrightarrow R$ is one and then

$$\ell_R(H_i^R(x_1^q, ..., x_n^q; T/(z))) = \ell_{R^q}(H_i^R(x_1^q, ..., x_n^q; T/(z))) =$$

$$\ell_{R^q}(H_i^{R^q}(x_1^q, ..., x_n^q; T/(z))).$$

The last equality holds because the last two Kozsul homology groups are isomorph as R^q-modules.

We will prove that $\lim_{q\to+\infty}\delta_R(x_1^q, ..., x_n^q; T/(z)) = +\infty$. In fact, since $\{x_1^q, ..., x_n^q\} \subseteq T$ is a system of parameters for the $R-$module $T/(z)$, we know that

$$\chi(x_1^q, ...x_n^q; T/(z)) = q^n\chi(x_1, ..., x_n; T/(z))$$

(see Corollary 5.2.4 [**34**]) and by previous comments,

$$\chi(x_1, ..., x_n; T/(z)) = e((x_1, ..., x_n), T/(z)) > 0,$$

because $\dim R > 0$ (if $\dim R = 0$ then R is a field and the DSC is trivial). Besides, by the previous corollary we know that there is a constant c such that $\ell_R(H_i(x_1^q, ..., x_n^q; T/(z))) < cq^{n-i}$ for each $i = 1, ..., n$ and each q. Combining this we get the following estimate

$$\delta_R(x_1^q, ..., x_n^q; T/(z)) = q^n e + \sum_{i=2}^{n}(-1)^{i+1}\ell_R(H_i^R(x_1^q, ..., x_n^q; T/(z)) \geq$$

$$q^n e - \sum_{i=2}^{n} cq^{n-i},$$

where $e = e(x_1^q, ..., x_n^q; T/(z)) > 0$. Let's write $f(q) := q^n e - \sum_{i=2}^{n} cq^{n-i}$. Then, the polynomial $f(q) \to +\infty$, when $q \to +\infty$, because it has positive leading coefficient. Therefore, $\delta_R(x_1^q, ..., x_n^q; T/(z)) = +\infty$. Let's fix $b = p^h > 0$. Then by the proof of Theorem 6.1 $R^b \hookrightarrow T/(z)$ splits. Denote by $\rho_1 : T/(z) \to R^b$ a splitting R^b-homomorphism. Finally, the Frobenius homomorphism $F_b : R \to R^b$ sending $x \to x^b$ is an isomorphism of rings.

Hence, we can define $\rho : T/(z) \to R$, by $\rho(x) := F_b^{-1}(\rho_1(x^b))$. Clearly, $\rho(1) = 1$ and if $x \in T/(z)$ and $r \in R$ then

$$\rho(rx) = F_b^{-1}(\rho_1((rx)^b)) = F_b^{-1}(\rho_1(r^b x^b)) =$$
$$F_b^{-1}(r^b \rho_1(x^b)) = F_b^{-1}(r^b) F_b^{-1}(\rho_1(x^b)) = r\rho(x).$$

So ρ is $R-$linear. In view of that $R \hookrightarrow T/(z)$ splits. $\square$

CHAPTER 5

The DSC for Some Quadratic Extensions

1. The DSC for Some Radical Quadratic Extensions

In this section we will prove the DSC for finite extensions generated by two elements satisfying radical quadratic equations in the setting of a UFD R, with $\text{char}(K(R)) \neq 2$ and when these satisfied quadratic equations and assuming that their coefficients satisfy a couple of arithmetical conditions.

In this section, let R be an integral domain and let $L = K(R)$ be its field of fractions. Let S be a module finite extension of R such that S is generated by two elements s_1 and s_2 that satisfy monic polynomial f_1 and f_2, respectively. Without loss of generality we can suppose that the degrees of these polynomials are greater that one, since in the case that one of the polynomials has degree one, it would imply that the corresponding root s_i belongs to R and therefore S is just generated by one element as $R-$module.

But, if R is a UFD, then it is easy to see that S is isomorphic to a quotient of a ring of the form $C = R[x]/(f(x))$, for a monic polynomial f, by an ideal J of height zero. Thus, J would be contained in some of the minimal primes of C, which are exactly principal ideals generated by the classes of the (monic) prime factors of $f(x)$ in the UFD $R[x]$. Hence, we can assume that J is, in fact, one of these primes, due to the fact that it is enough to find a retraction from a nonzero quotient of S to R. In conclusion, we can assume that S itself has the same form of C. But, it is elementary to see that C is a $R-$free module and therefore the extension splits.

Set

$$T = R[T_1, T_2]/I,$$

where $I = (f_1(T_1), f_2(T_2))$, for monic polynomials in $R[T_1, T_2]$, and let $\varphi :=$ $T \to S$ be the R-homomorphism sending T_1 to s_1 and T_2 to s_2. Let's denote with small letters the classes in T of the capital letters. It is easy to see that T is a free R-module because $T \cong R[T_1]/(f_1) \otimes_R R[T_2]/(f_2)$. In fact, an R-basis for T consists of monomials of the form $t_1^{d_1} t_2^{d_2}$, where $0 \leq d_i < \deg f_i$.

If Q is the kernel of φ then $S \cong T/Q$. Besides the Krull dimensions of the rings S and T coincide with the dimension of R because both extensions are finite over R and in particular integral (see Going up and its incomparability property, [**10**, Proposition 4.15 and Corollary 4.14]). Hence, the height of Q

must be zero, that means that Q is contained in some of the minimal primes of T.

In the next lemma we shall characterize the case, where T is an integral domain. This is important in the sense that if T is an integral domain then necessarily $Q = (0)$ and therefore $S = T$ is a direct summand of R, since T is a free R-module. In the following lemma i and j are indexes between 1 and 2, with $i \neq j$.

LEMMA 1.1. *Let R, T, f_1, f_2 be as above and let $E_i := L[T_i]/(f_i)$ and $F_j := E_i[T_j]/(f_j)$. Then T is an integral domain if and only if there is a i such that both E_i and F_j are fields. That is, if and only if f_i is irreducible in $L[T_i]$ and f_j are irreducible in $E_i[T_j]$.*

PROOF. First, note that $L \otimes_R T \cong L[T_1, T_2]/I \cong E_i[T_j]/(f_j) = F_j$ and the natural homomorphism $\mu := T \hookrightarrow L \otimes_R T$ is an injection because T is a torsion free R-module, since R is an integral domain and T is a free module. Therefore T is a subring of F_j, and if F_j is a field (in fact if it is an integral domain), then T is an integral domain. That gives the "only if" part of our lemma.

Conversely, let's assume that T is an integral domain and for the sake of contradiction that either E_i or F_j is not a field. In the first case, there are monic polynomials of positive degree g_1 and g_2 in $L[T_i]$ such that $f_i = g_1 g_2$ and $deg(g_s) < deg(f_i)$. Now, let $\alpha \in R \setminus \{0\}$ be a common denominator of the coefficients of g_1 and g_2. Then the equality $\alpha^2 f_i = (\alpha g_1)(\alpha g_2)$ in $R[T_i]$ implies that αg_i are zerodivisors in T. Besides, $\alpha g_i = \alpha t_i^{deg(g_i)} + ... \neq 0$ because g_i, written in the former R-basis of T, has some of the coefficients different from zero. Therefore T is not an integral domain, which is a contradiction. Then we can assume that E_i is a field.

Again, suppose for the sake of contradiction that f_j is reducible over $E_i[T_j]$. That is, $f_j = h_1 h_2$, where $h_1, h_2 \in E_i[T_j]$ are monic polynomials such that $0 < deg(h_n) < deg(f_j)$. Then, the difference $f_j - h_1 h_2$ is the zero polynomial in $E_i[T_j]$. Now, choose $H_1, H_2 \in L[T_i, T_j]$ such that $\psi(H_s) = h_s$, where $s \in \{1, 2\}$ and $\psi := L[T_i, T_j] \to E_i[T_j]$ is the natural homomorphism induced by the projection $L[T_i] \to E_i$. In fact, we can choose each H_i (considered as a polynomial in $(L[T_i])[T_j]$) such that each of its coefficients in $L[T_i]$ is a polynomial in T_i with degree smaller than $deg(f_i)$). Therefore, there exists a polynomial $H_3 \in L[T_1, T_2]$ such that $f_j - H_1 H_2 = H_3 f_i$. Finally, choose some nonzero element $c \in R$ such that $cH_r \in R[T_1, T_2]$, for $r = 1, 2, 3$. Thus $cH_1 cH_2 = c^2 f_j - c(cH_3) f_i \in I \subseteq R[T_1, T_2]$ and by construction the classes of cH_1 and cH_2 in T are nonzero, because some of the coefficients of the monomial R-base of T is not zero. In conclusion, T is not an integral domain, a contradiction. $\square$

COROLLARY 1.2. *Let R be an u UFD with* $\mathrm{char}(L) \neq 2$. *Assume that $f_i = T_i^2 - a_i$ are irreducible polynomials in $L[T_i]$ for $i = 1, 2$. If $T = R[T_1, T_2]/(f_1, f_2)$ is not an integral domain then there exist $c, d, u \in R \smallsetminus \{0\}$ such that $a_1 = d^2u, a_2 = c^2u$ and c, d are coprime.*

PROOF. By the symmetry and Lemma 1.1, there exists some such that f_2 is reducible in $E_1[T_2]$. But in our case this is equivalent to saying that f_2 has a root in E_1, say $e = e_1 + e_2\overline{T_1} \in E_1$, where $e_1, e_2 \in L$ and $\overline{T_1}^2 = a_1$. So

$$a_2 = e^2 = (e_1^2 + e_2a_1) + 2e_1e_2\overline{T_1},$$

and then $a_2 = e_1^2 + e_2^2a_1$ and $2e_1e_2 = 0$. But $\mathrm{char}L \neq 2$, then $e_1e_2 = 0$. If $e_2 = 0$, then $a_2 = e_1^2$ and $f_2 = (T_2+e_1)(T_2-e_1)$ a contradiction. Thus, $e_1 = 0$ and $a_2 = e_2^2a_1$. Write $e_2 = e/d$, where $c, d \neq 0$ are coprimes. So $d^2a_2 = e^2a_1$, but d^2 does not divide c^2, therefore d^2 divides a_1 and then there is a $u \in R$ such that $a_1 = d^2u$. Replacing in the forming equation we get $d^2a_2 = c^2d^2u$, and dividing by $d^2 \neq 0$, we get $a_2 = c^2u$, which proves our corollary. □

LEMMA 1.3. *Let R be a UFD; $B := R[X, Y]; u, c, d \in R \smallsetminus \{0\}; f_1 = X^2 - d^2u; f_2 = Y^2 - c^2u$ irreducible polynomials. Set $I = (f_1, f_2) \subseteq B$. Assume that $\{c, d\}$ is a regular sequence. Then the minimal prime ideals of I are $P_r = (f_1, f_2, dY + (-1)^rcX, XY + (-1)^rcdu)$, for $r = 0, 1$.*

PROOF. First, note that $f = T^2 - u$ has no roots in R, because f_1 has no roots in R. Therefore, f is irreducible in $R[T]$, and then (f) is a prime ideal, since R is a UFD. Define $\psi_r : B \to R[T]/(f)$ as the unique R-homomorphism sending $X \to dt$ and $Y \to (-1)^{r+1}ct$. We prove that $ker(\psi_r) = P_r$. In fact,

$$\psi_r(f_1) = d^2t^2 - d^2u = d^2u - d^2u = 0;$$
$$\psi_r(f_2) = c^2t^2 - c^2u = 0;$$
$$\psi_r(dY + (-1)^rcX) = d(-1)^{r+1}ct + (-1)^rcdt = 0;$$
$$\psi_r(XY + (-1)^rcdu) = (-1)^{r+1}cdt^2 + (-1)^rcdu = 0.$$

Conversely, it is a well known fact that over the ring of polynomials with coefficients in a commutative ring there exists division algorithm if the divisor is a monic polynomial. This justify the following procedure: let $h(X, Y)$ be a polynomial in $\ker(\psi_r)$, then there exists $Q(X, Y) \in B = (R[Y])[X]$ and $q_0(Y), q_1(Y) \in R[Y]$ such that $h(X, Y) = (X^2 - d^2u)Q(X, Y) + q_1(Y)X + q_0(Y)$. Besides, there exist $q_3(Y) \in R[Y]$ and $q_4, q_5 \in R$ such that $q_0(Y) = (Y^2 - c^2u)q_3(Y) + (q_4Y + q_5)$. Now, it is easy to see that there exists polynomials $q_6(Y), q_7(Y) \in R[Y]$ and constant $b_1, b_2, \in R$ such that

$$q_1(Y)X = (XY + (-1)^rcdu)q_6(Y) + q_7(y) + b_1X + b_2.$$

Moreover, we can divide $q_7(Y)$ by f_2 a getting a representation $q_7(Y) = q_8(Y)f_2 + q_9(Y)$. Replacing all of these equations in the first one, we see that there exists polynomials $Q_1(X,Y) \in B; Q_2(Y), Q_3(Y) \in R[Y]$ and $Q_5, Q_6, Q_7 \in R$ such that $h(X,Y) = Q_1(X,Y)f_1 + Q_2(Y)(XY + (-1)^r cdu) + Q_3(Y)f_2 + Q_4Y + Q_5X + Q_6$. Now, $0 = \psi_r(h) = Q_4(-1)^{r+1}ct + Q_5dt + Q_6$, and since $R[T]/(f)$ is a R-free module with basis $\{1,t\}$, we get $(-1)^{r+1}Q_4c + Q_5d = 0$, and $Q_6 = 0$. But $\{c,d\}$ is a regular sequence and then Q_5 is a multiple of c. Thus, there exists $w \in R$ such that $Q_5 = wc$. Replacing in the equation $(-1)^r Q_4 c = Q_5 d$ and dividing by c, we get $(-1)^r Q_4 = wd$ and so $Q_4 = w(-1)^r d$. In conclusion, $Q_4Y + Q_5X = w((-1)^r dY + cX)$, and then $h \in P_r$.

Finally, P_r is a prime ideal because B/P_r is isomorphic to the integral domain $R[T]/(f)$. On the other hand, in order to prove that P_0 and P_1 are the minimal primes of I let's consider any other minimal prime ideal $Q \in \operatorname{Spec}(B/I)$. Computing in B/I we get that $c^2f_1 - d^2f_2 = (cx-dy)(cx+dy) \in Q$, so $cx - dy \in Q$ or $cx + dy \in Q$. In the first case we deduce that $-x(cx-dy) = dxy - cx^2 = dxy - cd^2u = d(xy - cdu) \in Q$. Besides, $y(cx-dy) = cxy - dy^2 = cxy - dc^2u = c(xy - dcu)$. Assume by the sake of contradiction that $(xy - cdu) \notin Q$, then $c, d \in Q$ and thus $x^2 = (f_1 + d^2u), y^2 = f_2 + c^2u \in Q$, or equivalently, $x, y \in Q$. Now, if $(c,d) = R$, then $Q = B/I$ a contradiction. Therefore we can assume that $(c,d) \subsetneq R$. Since $x \notin P_1/I$ because every monomial in any generator of P_1, which is a multiple of X, is X^2, or XY, or cX, but $X \notin (X^2, XY, cX) \subsetneq B$, because $1 \notin (X,Y,c)$, so the monomial X cannot appear as an algebraic combination of the generators of P_1. In conclusion, $P_1/I \subsetneq (x,y,c,d) \subseteq Q$ and so Q is not minimal, a contradiction. Then, $(xy - cdu) \in Q$ and so $P_1/I \subseteq Q$, which is equivalent to $P_1/I = Q$ due to the minimality of Q.

In the second case, $cx + dy \in Q$ and we verify, as in the previous case, but multiplying the element $cx + dy$ by x and y, respectively, that $yx + cdu \in Q$, so $P_0 = Q$, finishing the proof. □

COROLLARY 1.4. *Let $R \hookrightarrow S$ be a finite extension of commutative rings such that R is an UFD, char$(L) \neq 2$ and S is generated by two elements $s_1, s_2 \in S$ satisfying monic radical quadratic polynomials $f_1 = T_1^2 - a_1$ and $f_2 = T_2^2 - a_2$, respectively. Then $R \hookrightarrow S$ splits.*

PROOF. By the Going Up, there exists a prime ideal $Q \subseteq S$, such that $Q \cap R = 0$. Therefore we can replace S by S/Q without changing the hypothesis and clearly it is enough to find a retraction from S/Q. In conclusion, we can assume that S is a domain.

As seen before S is isomorphic as R-algebra to T/J, where $T = R[T_1, T_2]/I$; $I = (f_1, f_2)$ and $\operatorname{ht}(J) = 0$. Now, $f_1 \in R[T_i]$ is irreducible for $i \in \{1,2\}$, otherwise, f_i would have a root in R and then S would be generated by just one element as R-module, and by the comment at the beginning of this section the extension splits on that case. If T is an integral domain then $J = 0$ and so $S = T \cong R^4$ is an R-free module and in this case it is clear that there exists a retraction.

On the other hand, if T is not a domain then, by Corollary 1.2 there exist $c, d, u \in R \setminus \{0\}$ such that $a_1 = d^2u$ and $a_2 = c^2u$ and c, d are coprimes, which implies that $\{c, d\}$ is a regular sequence. Besides, by Lemma 1.3, $\min(T) = \{P_0/I, P_1/I\}$ and then $J \subseteq P_r/I$ for some $r \in \{1, 2\}$. Hence if $\alpha : T/J \to T/(P_r/I)$ is the natural map, $\psi'_r : T/(P_r/I) \to R[T]/(T^2 - u)$ is the R-isomorphism induced by ψ_r as in the proof of

Lemma 1.3 and $\pi_1 : R[T]/(T^2 - u) \cong R^2 \to R$ is the projection in the first component, then a retraction is given by $\rho := \pi_1 \circ \psi'_r \circ \alpha$. This proves our corollary. □

Corollary 1.5. *Let $R \hookrightarrow S$ be a finite extension of commutative rings such that R is an UFD, $char(L) \neq 2$, 2 is a unit in R and S is generated by two elements $s_1, s_2 \in S$ satisfying monic quadratic polynomials $f_1 = T_1^2 + b_1T_1 + c_1$ and $f_2 = T_2^2 + bb_2T_2 + c_2$, respectively. Then $R \hookrightarrow S$ splits.*

Proof. We can reduce the proof to the radical quadratic case as in the previous Corollary, because it is easy to verify, by means of a elementary "completing the squares" argument that the following $R-$homomorphism is an isomorphism. We define

$$T' = R[U, V]/(U^2 + b - a^2/4, V^2 + c - d^2/4),$$

and

$$R[X, Y]/(X^2 - aX + b, Y^2 - cY + d),$$

and $\psi : T' \to T$ sending U to $X - a/2$ and V to $Y - c/2$.

In conclusion, S is isomorphism to a quotient of T' and so we can choose as generators of S as $R-$module elements satisfying monic radical quadratic equations, i.e., $s_1 = \overline{U}$ and $s_2 = \overline{V}$. Therefore by the previous Corollary the extension splits. □

2. The DSC for Some Nonradical Quadratic Extensions

Now, we prove a similar result for non-radical quadratic extensions by assuming two arithmetical conditions, one of them involving the discriminant of one of the quadratic polynomials.

Theorem 2.1. *Let R be an UFD and $R \hookrightarrow S$ a finite extension such that S is minimally generated by $s_1, s_2 \in S$. Assume that $f(s_1) = 0 = g(s_2)$, where $f(x) = x^2 - ax + b$ and $g(y) = y^2 - cy + d$ for some $a, b, c, d \in R$. If $\gcd(2, c) = 1$ and $a^2 - 4b$ is square free then $R \hookrightarrow S$ splits.*

In order to prove this Theorem we need the following lemma:

Lemma 2.2. *Let R be a UFD such that $2 \neq 0$, and $T = R[x, y]/(f(x), g(y))$, where $f(x) = x^2 - ax + b$ and $g(y) = y^2 - cy + d$ for some $a, b, c, d \in R$. Suppose* *that* $\gcd(2, c) = 1$ *and* $a^2 - 4b \neq 0$ *is square free. If* T *is not an integral domain, then there exists* $e \in R$ *such that* $(c \pm ae)/2 \in R$ *and the minimal primes of* T *are* $P_1 = (g(y), y - ex - (c - ae)/2)$ *and* $P_2 = (g(y), y - ex - (c + ae)/2)$.

PROOF. Since f is irreducible, because $u = a^2 - 4b$ is square free and assuming that T is not a domain, then, by Lemma 1.1, g is reducible on $L[y]$, where $L = K(R)[x]/(f(x))$. We can assume that $L \cong K(u^{1/2})$, for $u = a^2 - 4b$ and $K = K(R)$. Therefore $g(y)$ has a root $h = \alpha + \beta u^{1/2} \in L$, but one checks directly that the conjugate $\overline{h} = \alpha - \beta u^{1/2}$ is also a root of g, i.e. $g = (y - h)(y - \overline{h})$. By comparing coefficients we get $\alpha^2 - \beta^2 u = d$ and $c = 2\alpha$, so $4d = c^2 - 4\beta^2 u$. Let's write $\beta = q/r$, for $q, r \in R$ such that $\gcd(q, r) = 1$. From the last equation we get $4r^2 d = r^2 c^2 - 4q^2 u$, then $4(r^2 d + q^2 u) = r^2 c^2$. It implies that $4 \mid r^2 c^2$, but $\gcd(2, c) = 1$, therefore $4 \mid r^2$ and so $2 \mid r$. Write $r = 2t$, for some $t \in R \setminus \{0\}$. Hence, after canceling 4 in the last equation we get $4t^2 d + q^2 u = t^2 c^2$ or, equivalently, $t^2(c^2 - 4d) = q^2 u$. From this, it follows that $t^2 \mid q^2 u$, which implies $t^2 \mid u$, because $\mathrm{mcd}(t, q) = 1$. But, u is square free, therefore, t is a unit and then we can assume that $q/t \in R$. Defining $e = q/t$, we get

$$c^2 - 4d = e^2(a^2 - 4b). \tag{2.1}$$

We will prove that $2 \mid (c \pm ae)$. In fact, suppose that $2 = \prod p_i^{\gamma_i}$; and suppose $c + ae = \prod p_i^{\alpha_i}$; and that $c - ae = \prod p_i^{\beta_i}$ are the corresponding factorizations, where $\alpha, \beta, \gamma \geq 0$ (we can write in every expression the same primes because we allow the (unique) exponents to be zero). We shall see that $\gamma_i \leq \min(\alpha_i, \beta_i)$, for all i. From (2.1) it holds that

$$((c - ea)/2)((c + ea)/2) = d - e^2 b \in R,$$

this implies that $2\gamma_i \leq \alpha_i + \beta_i$, since $(c - ea)(c + ea)/2^2$ belongs to R. By the sake of contradiction suppose that there is j such that $\gamma_j > \min(\alpha_j, \beta_j)$. Without loss of generality, we may assume $\alpha_j = \min(\alpha_j, \beta_j)$. Hence, $\gamma_j \leq \beta_j$, otherwise $2\gamma_j > \alpha_j + \beta_j$, which is a contradiction. Therefore, $p_j^{\gamma_j} \mid (c - ae)$, then $p_j^{\gamma_j} \mid (c - ae) + 2ae = c + ae$, which means $\gamma_i \leq \alpha_i$. In conclusion, $\gamma_i \leq \min(\alpha_i, \beta_i)$ for all i, so $2 \mid (c \pm ae)$. Let $g_1 = y - ex - (c - ae)/2$ and $g_2 = y - ex - (c + ae)/2$. Using $((c - ea)/2)((c + ea)/2) = d - e^2 b$ we see by direct computation that

$$g_1 g_2 = e^2 f + g. \tag{2.2}$$

Now, (2.2) implies $P_1 = (g_1, f)$ and $P_2 = (g_2, f)$. Besides,

$$R[x, y]/(f(x), g_1) \cong R[x]/(f(x)) \cong R[x, y]/(f(x), g_2),$$

because in both cases we can eliminate the variable y using g_1 and g_2, respectively (for example the first isomorphism send y to $ex + (c - ae)/2$). Hence, P_1

and P_2 are prime ideals of T, because $R[x]/(f(x))$ is an integral domain. Finally, they are minimal primes because by (2.2) any prime ideal $P \in \mathrm{Spec}(T)$ should contain g_1 or g_2 and thus $P = P_1$ or $P = P_2$. □

Now, we prove Theorem 2.1.

PROOF. By previous results we know that $S \cong T/J$, where

$$T = R[x,y]/(f(x), g(y))$$

and $\mathrm{ht}J = 0$. Besides, $f(x) \in R[x]$ is irreducible, because otherwise there would exist $a, b \in R$ with $f(x) = (x-a)(x-b)$, so $s_1 = a$ or $s_2 = b$, which implies $S = R[s_2]$, contradicting the fact that s_1, s_2 generate S minimally as an $R-$module.

On the other hand, by the previous Lemma $J \subseteq P_j$, for some $j = 1, 2$. Finally, we get the desired retraction $\rho : S \to R$ as the composition of the following natural chain of $R-$homomorphisms

$$S = T/J \to T/P_j \xrightarrow{\varphi} R[x]/(f(x)) \to R \oplus R\overline{x} \xrightarrow{\pi_1} R,$$

where φ is the $R-$homomorphism defined by $\varphi(x) = x$ and $\varphi(y) = g_j - y$. □

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