The role of blending in mathematical invention

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Abstract

We model the mathematical process whereby new mathematical theories are invented. Here we explain the use of conceptual blending for this purpose, and show examples to illustrate the process in action. Our longer-term goal is to support machine and human mathematical creativity.

Introduction

We are concerned with creativity in mathematics: creativity as evinced by human and artificial mathematicians, individually and collectively.

Work on *conceptual blending* has been much influenced by Fauconnier and Turner (1998, 2002). More recently, the centrality of conceptual blending to creativity has been stressed by Turner (2014), where he writes:

> . . . the human spark comes from our advanced ability to *blend* ideas to make *new* ideas. Blending is the origin of ideas. (Turner, 2014, p 2)

The claim is that blending in this sense is a general human cognitive ability, and as such applies to mathematics, as much as to art, poetry, music and so on (see for example Turner (2005)).

The place of mathematics and the sciences among creative endeavours has been stressed by the literary critic George Steiner:

> It is in mathematics and the sciences that the concepts of creation and of invention, of intuition and of discovery, exhibit the most immediate, visible force.
>
> Steiner (2001, p 145)

Blending involves recognising features common to mathematical concepts, even when expressed in different terminology. The role of mathematical analogy in creative mathematics is well expressed by Weil (1960), and a general plea for analogical reasoning within science in (Arbib and Hesse, 1986).

We are investigating *computational* accounts of mathematical creativity, taking conceptual blending as a key ingredient. The work of Goguen (1999, 2005) has provided a general framework for comparison of conceptual spaces, and computation of blends. This enables the use of richer representation formalisms, and so is closer to contemporary mathematics than previous computational realisations of blending, such as in Pereira (2007).

This paper deals with the creative process in mathematics, as modelled along the lines above. We focus on the use of blending within a single process, searching for blends satisfying some evaluation criteria, from the starting point of some given conceptual spaces.

While cognitive issues are important to us, this paper is focused on issues in representation and representation change; there are however brief comments on cognition in the conclusions.

We start by providing some background, followed by an example to illustrate the components involved in our approach. A historical example based on Georg Cantor's work follows. The most extended example was carried out by a pure mathematician (D. Gómez-Ramírez), working in a domain close to his own; in this case, the blend mechanism threw up some unexpected properties, which provoked new work by the mathematician.

Subsequently we give some more speculative thoughts on where this work can go in the future, by considering Galois theory as a test-bed. Finally, we discuss the evaluation of work along these lines, and give some conclusions.

Background

Blending in Mathematics

Lakoff and Núñez (2000) are among the first to present a cognitive account of the origin and development of mathematical ideas,[1] arguing against the "romance of mathematics" in which mathematics is presented as an ever-increasing set of universal, absolute, certain truths which exist independently of humans. They present the thesis that human mathematics is grounded in bodily experience of a physical world, and mathematical entities inherit properties which objects in the world have, such as being consistent or stable over time. Exploring the physical world of object collection might lead to concepts like the empty collection and rules like "adding a collection of n objects to an empty

[1] This is lamented by Lakoff and Núñez, who claim that (prior to their work), "there was still no discipline of mathematical idea analysis from a cognitive perspective" (Lakoff and Núñez, 2000).

collection yields a collection with n objects". People then form grounding metaphors between the physical world and an abstract mathematical world, allowing us to project from everyday experiences onto abstract concepts, thus leading to the concept of zero and the axiom that $n+0 = n$. Lakoff and Núñez posit that blending different mathematical metaphors leads to more complex ideas (see also Alexander (2011)).

Alongside this account of mathematical cognition, mainstream contemporary mathematics has developed its own methodology and foundations, enjoying an exceptional place among scientific disciplines. Its methods, objects of study and sometimes astonishing results have widespread, if not universal, acceptance.

In conclusion, mathematics is a scientific discipline having not only a fundamental cognitive component, necessary in its development, but also possessing a collection of general principles and structures going beyond a particular school of thought. Among these general processes we want to highlight in this paper the importance that conceptual blending has in mathematics, incorporating both cognitive and mathematically specific aspects in order to create new mathematical concepts.

Terminology for conceptual blending

Our notion of conceptual blending is informed by Category theory, and highly influenced by Goguen's work on concepts (Goguen, 2005). In this paper we use the terminology below, and elucidate the terminology by means of a running example – discovering a version of the integers (in the sense of providing a partial approach to the genuine integers) using blending.

Conceptual spaces are partial and temporary representational structures which are constructed on the fly when talking about a particular situation, which are informed by the knowledge structures associated with a domain. These are influenced by Boden's idea of a concept space which is mapped, explored and transformed by transcending mapped boundaries (Boden, 1977), and form the input spaces to our blend.

As an example of two conceptual spaces, consider one as a theory NAT – a theory of the natural numbers, and FUNC – a theory of a total unary function with an inverse. We will refer back to these theories in this exposition.

(Many-sorted) First-order **Axioms** are the criteria which will be used here to delineate the conceptual spaces. The axiomatic method has been a fundamental aspect of mathematical research since Euclid, and various axiom changes have led to revolutions in mathematics. For instance, rejecting the parallel postulate opened up fascinating new areas of non-Euclidean geometry.

The precise formulations for NAT and FUNC can be found in Listings 1 and 2. Notice that these formulations obviously refer to partial representations of the genuine concepts employed by mathematicians. In the conceptual space with theory NAT, an example of an axiom is $\forall x.\neg\, 0 = s(x)$ – that is that zero is not a successor element. The conceptual space with theory FUNC has an axiom $\forall x.\ f(\mathit{finv}(x)) = x$.

Signature morphisms between conceptual spaces are mappings from the symbols of the source conceptual space into the symbols of the other conceptual space. For example NAT contains a function $\lambda x : \mathit{Nat}.\ s(x)$ that maps x to its successor, and FUNC contains a function defined over a set X that maps each element to an image $\lambda x : X.\ f(x)$. A theory G with a morphism to both NAT and FUNC might contain a function $\lambda x : N.\ \mathit{func}(x)$ that takes every number in some set N to its image under *func*. When we show a mapping we write this as

$$s \quad \leftarrow_{\phi(G,\text{NAT})} \quad \mathit{func} \quad \rightarrow_{\phi(G,\text{FUNC})} \quad f \qquad (1)$$

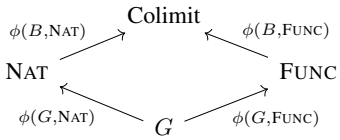
$$\mathit{Nat} \quad \leftarrow_{\phi(G,\text{NAT})} \quad N \quad \rightarrow_{\phi(G,\text{FUNC})} \quad X \qquad (2)$$

The mapping $\phi(G, \text{NAT})$ is a signature morphism from G to NAT. Note that associated types are also mapped.

Input Spaces refer to two or more conceptual spaces of interest.

Generic spaces are conceptual spaces that possess commonality between input spaces.

Colimits are conceptual spaces representing a blend of input spaces with respect to a given generic space and a set of signature morphisms. These are uniquely computed given a generic space and a set of morphisms. Here is a diagrammatic representation of such a computation in our example using theories NAT and FUNC :



The conceptual space represented by the Colimit is often referred to as the *blend*.

Internal Evaluation constitutes a variety of techniques to determine whether a computed colimit is viable as a conceptual space. In our example, since the conceptual spaces are mathematical theories, we can exploit the notion of consistency. This is a way of evaluating whether a blend is not only creative, but also valid. In the example of theories NAT and FUNC, the computed blend is inconsistent due to the emergent axioms in the computed colimit. The only type existing within the colimit is from now on referred to as $\mathbb{Z}$ to distinguish it from the natural numbers. Notice that in the colimit it holds that:

$$\forall x : \mathbb{Z}.\neg\ \mathit{zero} = s(x)$$

$$\forall x : \mathbb{Z}.\ s(\mathit{sinv}(x)) = x\ .$$

This is an inconsistency, as from the second axiom we see that there is an element for which 0 is the successor.

Weakening refers to the process of weakening the input theories by removing symbols or axioms. If we remove the axiom

$$\forall x : \mathit{Nat}.\neg\ \mathit{zero} = s(x)$$

then the resulting computed colimit contains a mathematical theory which is consistent.

Martinez et al. (2014) provides an algorithm to explore the space of blends resulting from given input spaces and a given generic space, where weakening is achieved by omitting axioms. The algorithm returns the blends which are consistent, and maximally so, among those in this space of

blends. This algorithm assumes that consistency of relevant theories can be checked, so is not always effective.

Running the blend refers to elaborating or completing a mathematical theory. Sometimes there are missing definitions which need to be discovered. For example in the new theory the following axiom appears

$$\forall x, y : \mathbb{Z}.\ s(x) + y \quad = \quad s(x + y)\ ,$$

but we also are interested in theorems such as

$$\forall x, y : \mathbb{Z}.\ \mathit{sinv}(x) + y \quad = \quad \mathit{sinv}(x + y)\ .$$

Finding suitable theorems is an example of running the blend, and from which it is possible to discover and prove theorems such as

$$\forall x, y : \mathbb{Z}.\ \mathit{sinv}(x) + s(y) \quad = \quad x + y\ .$$

Technologies

The approach explained above corresponds to Goguen's proposal (Goguen, 1999) for implementing blending, but slightly simplified (as in Kutz, Neuhaus, Mossakowski, and Codescu (2014)): we use the normal colimit construction, rather than $\frac{3}{2}$-colimits (both described in Goguen (1999)).

Additionally we assume that the conceptual spaces involved are given using a CASL specification (Astesiano et al., 2002) and that the morphisms are theorem preserving (i.e. map theorems to theorems). The reason for these assumptions is that in these cases it is well-known how to compute colimits: the colimit specification essentially corresponds to the disjoint union of the two target conceptual spaces except for not repeating the symbols given in the common source conceptual space. Moreover, we will be using the HETS system (Mossakowski, Maeder, and Lüttich, 2007) to compute such colimits. The code for the implemented examples in this paper is available on-line.[2]

The use of CASL specifications means that we deal with first-order logic; CASL is supported in the HETS system, and colimits here can be computed in the current implementation of HETS. Although higher-order logic (with Henkin semantics) is available in HETS (indeed in CASL) and the colimits are well-known to exist (because higher-order in this form is reducible to many-sorted first-order logic), it is worth noticing that the calculus of such colimits is not currently available in HETS. This restricts the formalisms that can be used directly for our purposes, where computation of colimits is central to our approach.

Blending and the infinite

Example Revisited – the Integers

As a first demonstration of the machinery involved in blending mathematical theories, we consider combining a theory of natural numbers with the concept of the inverse of a function to obtain the integers. Let us assume a simple partial axiomatisation of the natural numbers (without order axioms) as shown in Listing 1, and call this theory NAT. Now let us also define a simple theory which introduces the concept of a function with an inverse as shown in Listing 2, and call this theory FUNC.

[2] See: https://github.com/ewenmaclean/ICCC2015_hetsfiles

spec NAT =
sort *Nat*
ops *zero* : *Nat*;
s : *Nat* → *Nat*;
__+__ : *Nat* × *Nat* → *Nat*
∀ *x*, *y* : *Nat*
• *s*(*x*) = *s*(*y*) ⇒ *x* = *y*
• ¬ *zero* = *s*(*x*)
• *s*(*x*) + *y* = *s*(*x* + *y*)
• *zero* + *y* = *y*
end

Listing 1: A theory of the natural numbers without order

spec FUNC =
sort *X*
op *f* : *X* → *X*
op *finv* : *X* → *X*
∀ *x* : *X*
• *f*(*finv*(*x*)) = *x*
• *finv*(*f*(*x*)) = *x*
end

Listing 2: A theory with a function and its inverse defined

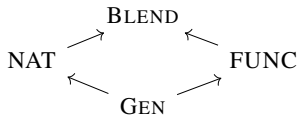
Identifying a Generic Space In order to incorporate the notion of blending here we want to be able to identify a "generic" component of each theory and compute the colimit. We can use the HDTP system (Gust, Kühnberger, and Schmidt, 2006; Schmidt, 2010) to discover a common theory and signature morphism between symbols in the two theories NAT and FUNC. The Generic theory GEN contains a sort *N* and a function *func*, and the morphisms from the Generic theory to NAT and FUNC are:

$$s \ \leftarrow_{\phi(G,\mathrm{NAT})}\ \mathit{func}\ \rightarrow_{\phi(G,\mathrm{FUNC})}\ f \qquad (3)$$

$$\mathit{Nat} \ \leftarrow_{\phi(G,\mathrm{NAT})}\ N\ \rightarrow_{\phi(G,\mathrm{FUNC})}\ X \qquad (4)$$

Here the successor function is identified in the mapping with the function in the theory FUNC.

Computing the Colimit The HETS system (Mossakowski et al., 2007) can then be exploited to find a new theory by computing the colimit:



This generates the theory shown in Listing 3 (for the sake of understanding it is used *p*, for predecessor, instead of *sinv*).

Removal of Inconsistencies This theory is automatically determined to be inconsistent due to the axioms

$$\forall x : \mathbb{Z}.\neg\ \mathit{zero} = s(x) \qquad (5)$$

$$\forall x : \mathbb{Z}.\ s(p(x)) = x \qquad (6)$$

```
spec SPEC =
    sort  N
    op    __+__ : N × N → N
    op    p : N → N
    op    s : N → N
    op    zero : N
    ∀ x, y : N • s(x) = s(y) ⇒ x = y
    ∀ x : N • ¬ zero = s(x)
    ∀ x, y : N • s(x) + y = s(x + y)
    ∀ y : N • zero + y = y
    ∀ x : N • s(p(x)) = x
    ∀ x : N • p(s(x)) = x
end
```

Listing 3: An inconsistent partial approach to the integers (without order)

Removal of the limiting axiom (5) from Listing 1 results in generating a blend theory which is very similar to what we understand to be the integers as shown in Listing 4.

```
spec SPEC =
    sort  N
    op    __+__ : N × N → N
    op    p : N → N
    op    s : N → N
    op    zero : N
    ∀ x, y : N • s(x) = s(y) ⇒ x = y
    ∀ x, y : N • s(x) + y = s(x + y)
    ∀ y : N • zero + y = y
    ∀ x : N • s(p(x)) = x
    ∀ x : N • p(s(x)) = x
end
```

Listing 4: A consistent partial approach to the integers (without order)

Running the Blend Running the blend refers to discovering definitions or adding axioms to flesh out the blend. In the example of the version in Listing 4, the definition of plus needs to be extended to understand how to calculate with the predecessor function:

$$p(x) + y = p(x + y)$$

from which theorems such as

$$p(x) + s(y) = x + y$$

can be proved.

Potential and actual infinity

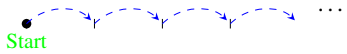
Some of the ideas of Lakoff and Núñez (2000) have been reworked by the authors, with increased emphasis on conceptual blending. In particular, the analysis of mathematical infinity, given in metaphorical form as the "Basic Metaphor of Infinity" (BMI) in Lakoff and Núñez (2000), is represented in blend form in Núñez (2005) as the "Basic Mapping of Infinity" (so, still "BMI").

We show here how this blend works out in our setting. The BMI suggests that the notion of completed infinity, in particular the possibility of transfinite numbers in the sense of Cantor, comes from a blend of the notion of completed, finite process with that of a potentially infinite and endless process.

Thus take two corresponding input spaces, given by CASL specifications **FinEnd** and **Inf** corresponding to the following diagrams

FinEnd:

Inf:



- **FinEnd:** Completed Iterative Processes are those that from some initial state, terminate in a final state after a finite number of state transitions. One such case is chosen.
- **Inf:** Infinite Iterative Processes are those that continue indefinitely to change state.

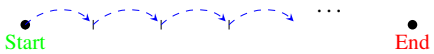
In both cases, the arrows indicate steps of the processes, and the process states are in a discrete linear order indicated by left-to-right order in the diagrams.

The generic space **Gen** simply identifies the start states, the notion of process step, and the linear ordering of states.

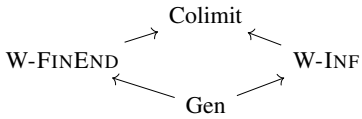
Now we can compute the blend of these spaces, which includes new features taken from both of the input spaces. This blend is *inconsistent*, for the following two reasons:

1. the number of states is finite (from **FinEnd**), and infinite (from **Inf**);
2. there both is an end state (from **FinEnd**) and is no end state (from **Inf**).

Search through the possibilities of weakening the input spaces by omitting as few axioms as possible among those involved in an inconsistency reveals the possibility of a structure with infinitely many states (from **Inf**) and an end state (from **FinEnd**). Computing the colimit from the weakened input spaces **W-FinEnd**, **W-Inf** gives a theory corresponding to this diagram:



Thus we have a blend as in the earlier examples:



Prime Ideals as a blend

Introduction

One of the most fundamental concepts of modern mathematics, which is the basis of commutative algebra and a seminal ingredient of the language of schemes in modern algebraic

geometry, is that of *prime ideal* (Grothendieck and Dieudonné, 1971; Eisenbud, 1995).

The terminology "prime ideal" relates to the older notion of "prime number". The initial aim of this work was to look for a blend between prime numbers (from the integers) and the ideals of a commutative ring, to see what would emerge. It turned out that *the blend process*, along with providing a definition for prime ideals, also *suggested an unexpected concept in the context of rings*, namely what will be called Containment Division Rings (CDR). In turn, this prompted questions and proofs about this concept – thus running the blend (space prevents description of this step in this paper).

We present a first blend involving weakening, followed by a second blend from fuller input spaces, where the emergent concept of CDR appears.

The first conceptual space

Let $(R, +.*, 0, 1)$ be a commutative ring with unity (see the formal definition and examples in Eisenbud (1995)). Now, R can be understood as the sort containing the elements of the corresponding commutative ring with unity. An ideal I is a subset of R satisfying the following axiom:

$$(\forall i, j \in I)(\forall r \in R)(i + (-j) \in I \wedge r * i \in I).$$

Let us define a unary relation (predicate) *isideal* on the set (sort) of subsets of $P(R)$ corresponding to this definition. Now, we define

$$\mathrm{Id}(R) = \{A \in P(R) : \mathit{isideal}(A)\}\,.$$

Ideals are "multiplied" using the following definition:

$$I \cdot_\iota J = \left\{ \sum_{k=1}^{n} i_k \cdot j_k : n \in \mathbb{N} \wedge i_1, \ldots, i_n \in I \wedge j_1, \ldots, j_n \in J \right\}.$$

In other words, $I \cdot_\iota J$ is the smallest ideal extending the set $\{i \cdot j : i \in I \wedge j \in J\}$.

The key property that we want to keep in the blend is the one saying that this operation $\cdot_\iota$ has a neutral element 1_ι, which can be seen as an additional notation for the ring. On the other hand, we want to see the containment relation $\subseteq$ as a binary relation over the sort $\mathrm{Id}(R)$.

Summarizing, our first conceptual space consists of sorts $R, \mathrm{Id}(R)$ and $P(R)$; operations $+, *, 0_R, 1_R, 1_\iota$ and $\cdot_\iota$; and the relations $\subseteq$ and *isideal*.

Let us denote this space by $\mathbb{I}$.

The second conceptual space

Let $\mathbb{Z}$ be the set of the integer numbers. Here, we choose any partial axiomatization of them including at least the fact that $(\mathbb{Z}, *, 1)$ is a commutative monoid. We define also an upside-down divisibility relation $\lfloor$ defined as $e \lfloor g := g|e$, i.e. there exists an integer c such that $e = c * g$. Let us define a unary relation *isprime* on $\mathbb{Z}$ as follows: for all $p \in \mathbb{Z}$, $\mathit{isprime}(p)$ holds if $p \neq 1$ and:

$$(\forall a, b \in \mathbb{Z})\ \big((ab \lfloor p) \rightarrow (a \lfloor p \vee b \lfloor p)\big).$$

Besides, we define the set (sort) of the prime numbers as

$$\mathit{Prime} = \{p \in \mathbb{Z} : \mathit{isprime}(p)\}$$

In the CASL language, we consider $\mathbb{Z}$ as the sort of the integer numbers, $*$ as a binary operation, *prime* as a predicate and $\lfloor$ as a binary relation, any of them defined over the sort $\mathbb{Z}$. We denote this conceptual space by $\mathbb{P}$.

The Generic Space

The generic space $\mathbb{G}$ consists of a set (sort) G with a binary operation $*_G$, a neutral element S and a binary relation $\leq_G$.

The Blending Morphisms

The morphism to $\mathbb{I}$ uses:
$\varphi(G) = \mathrm{Id}(R), \varphi(*_G) = *_\iota, \varphi(S) = 1_\iota$ and $\varphi(\leq_G) = \subseteq$;
the morphism to $\mathbb{G}$ uses:
$\delta(G) = \mathbb{Z}, \delta(*_G) = *, \delta(S) = 1$ and $\delta(\leq_G) = \lfloor$.

The Axiomatization of the Blending

A straightforward colimit construction based on the input and generic spaces above yields a consistent space with properties inherited both from the prime elements into the integers and from the ideals of commutative rings; one of the concepts is a notion of prime ideals, another is that of CDR.[3] Here we describe briefly a weakening of the given spaces that makes the resultant blend more generally applicable.

From the properties defining the integers we transfer into the blend only the fact that $\mathbb{Z}$ is a set with a binary operation $*$ having 1 as neutral element and $\lfloor$ as a binary relation, without taking into account its formal definition.

Now after computing the colimit, we obtain that any element $P \in G$ (i.e., an ideal of S) satisfies the predicate *isprime* if and only if

$$P \neq S \wedge (\forall X, Y \in G = \mathrm{Id}(S))(X \cdot_\iota Y \subseteq P \rightarrow (X \subseteq P \vee Y \subseteq P)).$$

Thus, the predicate *isprime* turns out to be the predicate characterizing the primality of ideals of S and the set (sort) Prime turns out to be the set of prime ideals of S.

Using the weakened input spaces, the blending space consists of the axioms assuring that S is a commutative ring with unity, G is the set of ideals of S, *isprime* is the predicate specifying primality for ideals of S and *Prime* is the collection of all prime ideals of S.

Implementation for prime ideals over CDR-s as a blend

In this section we construct the concept of prime ideal over a CDR as a blend of the conceptual space of ideals of a commutative ring with unity and the conceptual space of the former second conceptual space where the axiom defining the upside-down divisibility relation is restored.

It is worth mentioning again that the definition of CDR-s was obtained after doing this implementation and therefore it could be seen as a form of "creative" result coming from the blending process.

After computing the corresponding colimit in HETS and interpreting "RingElt" as the sort containing the elements of the ring S, the theory defining the blend corresponds to the axioms defining a CDR (*S*), the set of all its ideals (*Generic*), the set all its prime ideals (*SimplePrime*) and a primality predicate (*IsPrime*). We present in Listing 5 just the

[3] A ring R is a Containment Division Ring (CDR) if for all ideals I and J of R, $I \subseteq J$ if and only if J divides I (i.e. there exists an ideal U such that $I = U \cdot_\iota J$).

theory corresponding to the colimit (omitting details of ring axioms and ideal generation).

```
spec SPEC =
    sorts Generic, RingElt, SimplePrime, SubSetOfRing
    sorts SimplePrime < Generic,IdGeneric < SubSetOfRing
    ops 0, 1, S : RingElt
    op __*__ : RingElt × RingElt → RingElt
    op __+__ : RingElt × RingElt → RingElt
    op __x__ : Generic × Generic → Generic
    pred IsIdeal : SubSetOfRing
    pred IsPrime : Generic
    pred __isIn__ : RingElt × SubSetOfRing
    pred gcont : Generic × Generic
    pred __generates__ : RingElt × Generic
    ∀ I : SubSetOfRing • I ∈ Generic ⇔ IsIdeal(I)
    ∀ x : Generic • x x S = x
    ∀ x : Generic • S x x = x
    ∀ A, B : Generic
    • gcont(A, B) ⇔ ∀ a : RingElt • a isIn A ⇒ a isIn B
    ∀ x, y : RingElt • x + y = y + x
    %% and further ring axioms ...
    ∀ I : SubSetOfRing
    • IsIdeal(I)
      ⇔ ∀ a, b, c : RingElt
        • ((a isIn I ⇒ a isIn S) ∧ 0 isIn I)
          ∧ (a isIn I ∧ c isIn S ⇒ c ∗ a isIn I)
          ∧ (a isIn I ∧ b isIn I ∧ c isIn S ∧ b + c = 0
              ⇒ a + c isIn I)
    ∀ a : RingElt; A : Generic
    %% and axioms for generates and x ...
    ∀ x, y : Generic • gcont(x, y) ⇔ ∃ c : Generic • x = y x c
    ∀ p : Generic • p ∈ SimplePrime ⇔ IsPrime(p)
    ∀ p : Generic
    • IsPrime(p)
      ⇔ (∀ a, b : Generic
          • gcont(a x b, p) ⇒ gcont(a, p) ∨ gcont(b, p))
        ∧ ¬ p = S
end
```

Listing 5: Colimit for prime ideals over CDR-s

A Challenge Example for Blending

Computational Creativity via Blending

The examples shown thus far in the paper have been examples of blending in mathematics whose mechanisation has helped to identify some novel and unexpected results. The blending itself was a one-stage process where human input was required to identify the input concepts. A more ambitious aim of the approach of applying blending to the problem of computational creativity in mathematics, is to allow search to be done over multiple blends and for the *process* of blending to be controlled mechanically. In this section we describe very informally a mathematical domain that seems in some ways a natural candidate for a blending approach.

Galois Theory

Galois theory develops a relationship between a polynomial $p(x)$ with coefficients in some field F, the extension of K of F (written "K/F") containing all of the roots of $p(x)$ in the algebraic closure of F, and the group $\mathbf{Gal}(K)$ of automorphisms of K/F that fix the elements of F. The fundamental theorem of Galois theory states that there is a bijection between the subfields of K/F and the subgroups of $\mathbf{Gal}(K)$; namely, subgroups correspond to their fixed fields. Using this correspondence, properties of polynomials can be derived, most famously the fact that quintic polynomials cannot be solved by algebraic operations and the extraction of roots.

We do not propose to reconstruct much of the theory here, but note that already in this basic account there are several steps that seem compellingly "blend-like."

In the first place, for field extension, E is an extension of F if F is a subfield of E. We could derive the extension relationship from the input concepts E and F by "taking everything additional from E and adding it to F." This is made specific in the process of *adjoining* elements, which simply means to augment the field with all fractions of formal finite sums and products of the adjoined elements with coefficients in the base field.

Second, the notion of the *splitting field* of a polynomial, namely the special extension K/F containing all of the roots of $p(x)$. This could be formed conceptually by combining the concept "*the roots of a polynomial $p(x)$ with coefficients in a field F*" and the concept "*a field extension E/F formed by adjoining certain elements to F*."

As above, we could then form the concept of $\mathbf{Gal}(K)$ by blending at the conceptual level. This time, there would be several constituent pieces: "*the roots of a polynomial $p(x)$ with coefficients in a field F*," "*the splitting field of $p(x)$*," "*the group of automorphisms of a field extension E*," "*the automorphisms that fix F*."

Finally, assuming that we have built $\mathbf{Gal}(K)$ in this fashion, we would like to know some of its properties. Consider the claim that *elements of $\mathbf{Gal}(K)$ permute the roots of f*. This time, instead of being purely conceptual, we want to work at the *process* level, and consider before-and-after descriptions of the result of applying $\varphi \in \mathbf{Gal}(K)$ to some r with the property $p(r) = 0$. This is similar in some ways to the "Riddle of the Buddhist Monk", popularised by Koestler (1964), which is cited as an example of the power of blending.[4] However, this time the generic space is not a simple geometric machine, but rather an algebraic machine with several moving parts.

The proof of the claim is as follows. If $p(r) = 0$, then $\varphi p(r) = \varphi 0$. Since φ is an automorphism, $\varphi 0 = 0$; and furthermore φ distributes over the sums and products that make up the polynomial $p(x)$ and fixes its coefficients, therefore $\varphi p(r) = p(\varphi r)$. Chaining the equalities together, we have $p(\varphi r) = 0$.

[4] "A Buddhist monk begins at dawn one day walking up a mountain, reaches the top at sunset, meditates at the top for several days until one dawn when he begins to walk back to the foot of the mountain, which he reaches at sunset. Making no assumptions about his starting or stopping or about his pace during the trips, prove that there is a place on the path which he occupies at the same hour of the day on the two separate journeys."

In short, the proof is a fairly direct result of combining the definitions. Goguen (1992) suggests that "combination is colimit." Can we realise the proof through (one or several) colimit operations? And is there anything special about this proof? Apart from these more theoretical questions, the foregoing discussion raises the following technical issues:

Field Extension When reasoning about polynomials, it is useful to distinguish the three separate types – those of E, those of F and those of E/F as a supertype. Using blending machinery removes the distinction between these types.

Splitting Field Extension Theorem A challenging but creative step is to discover the theorem that extending F *only* with the roots of $f(x)$ forms a field.

Automorphisms As mentioned in the background section, currently there is no way of computing colimits if automorphisms are characterised in higher-order logic. An alternative specification, or an implementation of colimit computation for higher-order logic is needed.

Evaluation and Outlook

Review of the current offering

(a) We began the paper with the reconstruction of certain mathematical objects, showing the technical feasibility of the approach.

(b) The more advanced example at the centre of the paper illustrates how this sort of reconstruction relates to mathematical practice.

(c) A future-oriented example exposes some technical challenges, while suggesting that blending could offer a novel approach to computer mathematics.

Broader issues in evaluation

In addition to motivating a further investigation of the role blending can play in proofs, Galois theory, discussed above, is paradigmatic for other reasons. This discussion draws on the early 20th Century writings of Albert Lautman on the philosophy of mathematics and the subsequent interpretation of this work by Gilles Deleuze. It uses these ideas to propose an approach to embedding evaluation within the system itself.

Concerning the common features of Galois theory, class field theory, and the development of the universal covering surface in Riemann geometry, (Lautman, 2011, p. 126) writes:

> What is characteristic of the movement of the theories that will be considered here is the existence of an end conceived in advance as a term of the ascent.

This is reminiscent of our notion of internal evaluation that apply to the blend. To illustrate, let us briefly imagine how we would use blending techniques to move from porcupine+lion to the perfected *porculione*. Here, instead of field automorphisms that preserve mathematical structure and fix certain designated elements, we would look for mappings that preserve other properties that exist in the underlying domain. Porculiones would presumably have four feet, would be mammals, and would be omnivores; they should also be viable living creatures.

(Deleuze, 1994, pp. 227–228) follows Lautman in enthusiastically endorsing the Galoisian approach to mathematics:

> [T]he fact that an equation cannot be solved algebraically, for example, is no longer discovered as a result of empirical research or by trial and error, but as a result of the characteristics of the groups and partial resolvents which constitute the synthesis of the problem and its conditions (an equation is solveable only by algebraic means – in other words, by radicals, when the partial resolvents are binomial equations and the indices of the groups are prime numbers). The theory of problems is completely transformed and at last grounded, since we are no longer in the classic master-pupil situation where the pupil understands and follows a problem only to the extent that the master already knows the solution and provides the necessary adjunctions. For, as Georges Verriest remarks, the group of an equation does not characterise at a given moment what we know about its roots, but the objectivity of what we do not know about them. Conversely, this non-knowledge is no longer a negative or an insufficiency but a rule or *something to be learnt* which corresponds to a fundamental dimension of the object.

Although there is a commonality between blending and the Galoisian approach insofar as progressive refinement carries us toward a "perfected" conclusion, Deleuze's enthusiasm about the pedagogical situation would be significantly cooled here. It would seem, in many of our examples, that we only make progress "to the extent that the master already knows the solution and provides the necessary adjunctions."

However, this apparent infelicity may be less of a thick obstacle than it would initially appear. What seems to be most needed is a notion of a *question* inside the system. This would recover Lautman's basic thrust: "Scientific or not, every question has built in some assumptions about the form of the answer" (Larvor, 2011). In short, an experimental approach in which the system *asks* and *answers* questions would embed key aspects for evaluation in the system itself.

Future work

The idea of using blending to carry out steps in a proof would provide a useful training ground for further development. The primary problem is: If blending is the realisation of "combinatorial creativity" how will we avoid being swamped by the combinatorial explosion of possible things to combine? The first challenge is thus fitting different mathematical components together in a sensible manner. A related challenge would apply when modifying the system to selectively experiment with the rules it uses. The objective in this case would be for the system to learn to associate different (useful) techniques with different types of problems.

Conclusions and Remarks

The examples presented in this paper trace the development of the blending approach. The current paper begins with reconstructions, but also quickly shows how computed blends

can suggest new mathematical definitions and concepts of interest to practising mathematicians. The analysis offered here shows that this work is a building block that will be useful for future developments that are able to reason more flexibly about mathematical problems – and systematically find and propose new concepts and problems.

In future work, we will look more at the cognitive issues raised in this work. In particular, the use of *image schemas* can give a link between the computational and representational approach taken here, and the cognitive claims coming from authors such as Fauconnier and Turner, and Johnson. Here the work of Mandler and Canovás (2014) and Hedblom, Kutz, and Neuhaus (2014) gives an idea of how these underlying cognitive primitives can be expressed in logical form, and can thus play an explicit role in our modelling of creativity in mathematics.

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